April 7, 2021

Round 1: Arithmetic and Number Theory

1. For any positive integer n, let $S(n)$ denote the sum of the factors of n, including 1 and n. For example, $S(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$. Suppose that $S(6) + S(m) = \{18,19,16,15\}$. Find m.

2. The sum of a sequence of thirteen consecutive integers is $\{13,26,39,52\}$. Find the median of the sequence.

3. Four married couples and {five, six, seven, eight} additional unmarried people are traveling together. For lunch, 5 people will go to Arby's and the remaining people will go to Long John Silver's. Each married person will eat at the same location as his/her spouse. In how many ways can the eating locations be assigned?

Answers

1) ______________________________

2) ______________________________

3) ______________________________

April 7, 2021
Round II: Algebra I (Real numbers and no transcendental functions)

1 When Annabel performs a task by herself, it takes her 54 minutes. When Boris performs the same task by himself if takes him {38, 42, 46, 34} minutes. If Annabel and Boris work together, how long would it take them to complete the task? (Give your answer to the nearest minute. Do not include a unit.)

2. Find the value of the expression given below.

$$\{\frac{1}{\sqrt{2}+\sqrt{1}}+\frac{1}{\sqrt{3}+\sqrt{2}}+\frac{1}{\sqrt{4}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{121}+\sqrt{120}},$$
$$\frac{1}{\sqrt{2}+\sqrt{1}}+\frac{1}{\sqrt{3}+\sqrt{2}}+\frac{1}{\sqrt{4}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{144}+\sqrt{143}},$$
$$\frac{1}{\sqrt{2}+\sqrt{1}}+\frac{1}{\sqrt{3}+\sqrt{2}}+\frac{1}{\sqrt{4}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{100}+\sqrt{99}},$$
$$\frac{1}{\sqrt{2}+\sqrt{1}}+\frac{1}{\sqrt{3}+\sqrt{2}}+\frac{1}{\sqrt{4}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{169}+\sqrt{168}}\}$$

3. Two particles, *P* and *Q*, move in the same direction on a circle of radius 1 meter. At time $t = 0$, both particles are at point *A*. *P* moves at a constant speed of {12π, 11π, 10π, 9π} meters per second and *Q* moves at a constant speed of {5π, 4π, 3π, 2π} meters per second. There are times $t > 0$ when the particles coincide (meaning that they are in the same place). At the third such time, the length of the minor arc *AP* is $\frac{m\pi}{n}$ meters, where *m* and *n* are relatively prime positive integers. Find $m + n$.

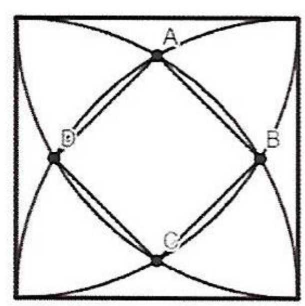
April 7, 2021

Round III: Geometry (figures are not to scale)

1. If $m\angle A = (6x+2y)°$ and $m\angle B = \{(15y)°, (3y)°, (6y)°, (5y)°\}$, angles A and B are congruent, and angles A and B are supplementary, find the value of x.

2. A rhombus has diagonals with lengths in the ratio of 2:1 and a perimeter of $\{10,30,40,50\}$. Find the area of the rhombus.

3. Four quarter circles of radius 1 are drawn in a unit square such that the center of each quarter circle is a vertex of the square as shown in the figure. The intersections of the quarter circles are connected to form square *ABCD*. The area of $ABCD$ can be expressed as $m-\sqrt{n}$ where m and n are positive integers. Find the value of $\{\frac{1}{2}m+3n, 2m-n, m+2n, 7m-4n\}$.



Answers

1) ______________________________

2) ______________________________

3) ______________________________

April 7, 2021
Round IV: Algebra II

1. Let $f(x) = x^3 + Ax^2 + Bx + C$, where A, B, C are constants. The graph of $y = f(x)$ is tangent to the x-axis at the point (1, 0) and passes through the point ({6,5,4,3}, 0). Find $|A| + |B| + |C|$.

2. Let $\log 2 = a$ and $\log 3 = b$. Then $\{\log_{250} 18, \log_{250} 12, \log_{250} 24, \log_{250} 54\} = \frac{pa+qb}{r-sa}$, where p, q, r and s are positive integers, and p and r are relatively prime. Find $p + 2q + 3r + 4s$. (Here, $\log x$ is understood to mean $\log_{10} x$.)

3. Let $f(x) = \left\{\frac{1}{4}, \frac{1}{3}, \frac{1}{4}, \frac{1}{3}\right\} x$ and $g(x) = x^3 + x$. One of the points of intersection of the graphs of $y = f(x)$ and $y = g^{-1}(x)$ is (p, q), where $p > 0$ and $q > 0$. Find $\{2p^2 + q^2, 2p^2 + q^2, p^2 + 2q^2, p^2 + 2q^2\}$.

Answers

1) ______________________________

2) ______________________________

3) ______________________________

April 7, 2021
Round V: Analytic Geometry

1. A line in the coordinate plane has equal x- and y-intercepts and passes through the point $\{(-4,-8),(-2,-6),(-3,-7),(-5,-1)\}$. Determine the area of the right triangle formed by the line and the coordinate axes.

2. A hyperbola has equation $x^2 - y^2 - 2ax + 4by + a^2 - 4b^2 - 1 = 0$, where a and b are constants. The asymptotes of the hyperbola have equations $\{x + y = 11, x + y = 10, x + y = 14, x + y = 11\}$ and $\{x - y = -1, x - y = 6, x - y = -2, x - y = 3\}$. Find $5a + 3b$.

3. Points A, B, and C have coordinates $(1, \{13,19,26,34\})$, (k, k^2), and $(2, 4)$ respectively. There are two possible values of k such that $4 < k^2 < \{13,19,26,34\}$ and the acute angle between $\overline{AB}$ and the line $x = k$ is congruent to the acute angle between $\overline{BC}$ and the line $x = k$. If the two possible values of k are k_1 and k_2, find the value of $|2k_1k_2|$.

Answers
1) ______________________________

2) ______________________________

3) ______________________________

April 7, 2021
Round VI: Trigonometry and Complex Numbers

1. Let A be the point (4, 3), B be the point (2, {5,6,7,8}), and O be the point (0, 0). Find the measure of angle AOB, rounding your answer to the nearest degree. (Do not include a unit in your answer.)

2. $i + 2i^2 + 3i^3 + \cdots + \{1004{,}1008{,}1012{,}1016\}i^{\{1004{,}1008{,}1012{,}1016\}} = A + Bi$, where A and B are real.
Find $2|A| + |B|$.

3. Let $t = \tan x$. Then, for all values of x apart from odd integer multiples of $\frac{\pi}{6}$,

$$\frac{\sin 6x}{1 + \cos 6x} = \frac{At^3 + Bt^2 + Ct + D}{Et^2 + Ft + G}$$

where the coefficients $A, B, \ldots, G$ are integers, some of them zero, and A and E are relatively prime. Find $\{2|A| + |B| + |C| + |D| + |E| + |F| + |G|, 3|A| + |B| + |C| + |D| + |E| + |F| + |G|, 2A^2 + B^2 + C^2 + D^2 + E^2 + F^2 + G^2, 3A^2 + B^2 + C^2 + D^2 + E^2 + F^2 + G^2\}$.

Answers
1) ______________________________

2) ______________________________

3) ______________________________

April 7, 2021
Team Round

1. How many positive integer bases a such that $a \leq 2021$ satisfy the inequality $105_a > 10_{6a+2021}$?

2. Find the area of the trapezoid with vertices (5, 6), (11,10), (16, 9), and (4, 1).

3. A circle with an area of π is inscribed in a right triangle of area 12. Find the length of the hypotenuse of the right triangle.

4. Let $f(x) = \frac{b}{9}x^2 - bx + 3b$, where b is a real constant. The remainder when $f(x)$ is divided by $x - 3b$ is 28. $f(2) = \frac{p}{q}$, where p and q are relatively prime integers and $q > 0$. Find $p + q$.

5. The circle $x^2 + y^2 - 10x - py + k = 0$ where $p > 0$ is tangent to both the x-axis and the line $y = 2\sqrt{2}x$. Find the value of kp^2.

6. Among complex numbers z that have the property that $|z - (8 + 6i)| = 3$, what is the maximum value of $|z|$?

Answers

Round 1

1. 5, 4, 3, 2
2. 1, 2, 3, 4
3. 71, 122, 203, 328

Round 2

1. 22, 24, 25, 21
2. 10, 11, 9, 12
3. 9, 11, 11, 9

Round 3

1. 13, 5, 10, 9
2. 5, 45, 80, 125
3. 10, 1, 8, 2

Round 4

1. 27, 23, 19, 15
2. 22, 21, 22, 24
3. 99, 38, 54, 22

Round 5

1. 72, 32, 50, 18
2. 34, 43, 42, 41
3. 15, 21, 28, 36

Round 6

1. 31, 35, 37, 39
2. 1506, 1512, 1518, 1524
3. 9, 10, 21, 22

Team Round

1. 1973
2. 39
3. 11
4. 61
5. 1250
6. 13