CSAML 2021

April 7, 2021
Round 1: Arithmetic and Number Theory

1. (1 point) For any positive integer n, let $S(n)$ denote the sum of the factors of n, including 1 and n. For example, $S(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$. Suppose that $S(6) + S(m) = \{18,19,16,15\}$. Find m. [Answer: $5, 4, 3, 2$]

Solution:
Since $S(6) = 1 + 2 + 3 + 6 = 12$, it follows that $S(m) = 6$. Note that $S(5) = 1 + 5 = 6$, so $m = 5$.

2. (2 points) The sum of a sequence of thirteen consecutive integers is $\{13,26,39,52\}$. Find the median of the sequence. [Answer: $1, 2, 3, 4$]

Solution:
This can be set up algebraically where n represents the first term as $n + n + 1 + \cdots + n + 12 = 13 \rightarrow 13n + 78 = 13$, so $n = -5$. The median would be the seventh integer in this sequence, which is 1.

3. (3 points) Four married couples and {five, six, seven, eight} additional unmarried people are traveling together. For lunch, 5 people will go to Arby's and the remaining people will go to Long John Silver's. Each married person will eat at the same location as his/her spouse. In how many ways can the eating locations be assigned? [Answer: $71, 122, 203, 328$]

Solution:
One way to solve this is to choose which married couples will go to Arby's and then choose the remaining people going to Arby's from the unmarried people. Since only five people will go to Arby's, up to two married couples can go. The three possibilities in this model are: zero married couples, $\binom{4}{0}\binom{5}{5}$; one married couple, $\binom{4}{1}\binom{5}{3}$; and two married couples, $\binom{4}{2}\binom{5}{1}$. Adding these options together gives $(1)(1) + (4)(10) + (6)(5) = 71$ possible ways.

CSAML 2021

April 7, 2021
Round II: Algebra I (Real numbers and no transcendental functions)

1. (1 point) When Annabel performs a task by herself, it takes her 54 minutes. When Boris performs the same task by himself if takes him {38, 42, 46, 34} minutes. If Annabel and Boris work together, how long would it take them to complete the task? (Give your answer to the nearest minute. Do not include a unit.) [Answer: 22, 24, 25, 21]

Solution:
Annabel's rate in tasks per minute is $\frac{1}{54}$. Boris's rate is $\frac{1}{38}$. So, their rate working together is $\frac{1}{54}+\frac{1}{38}$. Thus, the time taken to perform the task working together is $1/\left(\frac{1}{54}+\frac{1}{38}\right)=22$ minutes, to the nearest minute.

2. (2 points) Find the value of the expression given below.

$$\begin{aligned}&\{\frac{1}{\sqrt{2}+\sqrt{1}}+\frac{1}{\sqrt{3}+\sqrt{2}}+\frac{1}{\sqrt{4}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{121}+\sqrt{120}},\\ &\frac{1}{\sqrt{2}+\sqrt{1}}+\frac{1}{\sqrt{3}+\sqrt{2}}+\frac{1}{\sqrt{4}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{144}+\sqrt{143}},\\ &\frac{1}{\sqrt{2}+\sqrt{1}}+\frac{1}{\sqrt{3}+\sqrt{2}}+\frac{1}{\sqrt{4}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{100}+\sqrt{99}},\\ &\frac{1}{\sqrt{2}+\sqrt{1}}+\frac{1}{\sqrt{3}+\sqrt{2}}+\frac{1}{\sqrt{4}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{169}+\sqrt{168}}\}\end{aligned}$$

[Answer: 10, 11, 9, 12]

Solution:
Rationalizing the denominator of each fraction we get

$$\frac{\sqrt{2}-\sqrt{1}}{2-1}+\frac{\sqrt{3}-\sqrt{2}}{3-2}+\cdots+\frac{\sqrt{121}-\sqrt{120}}{121-120}$$

$$=\left(\sqrt{2}-\sqrt{1}\right)+\left(\sqrt{3}-\sqrt{2}\right)+\left(\sqrt{4}-\sqrt{3}\right)+\cdots+\left(\sqrt{120}-\sqrt{119}\right)+\left(\sqrt{121}-\sqrt{120}\right)$$

$$=\sqrt{121}-\sqrt{1}=11-1=10.$$

3. (3 points) Two particles, P and Q, move in the same direction on a circle of radius 1 meter. At time $t = 0$, both particles are at point A. P moves at a constant speed of {12π, 11π, 10π, 9π} meters per second and Q moves at a constant speed of {5π, 4π, 3π, 2π} meters per second. There are times $t > 0$ when the particles coincide (meaning that they are in the same place). At the third such time, the length of the minor arc AP is $\frac{m\pi}{n}$ meters, where m and n are relatively prime positive integers. Find $m + n$.
[Answer: 9, 11, 11, 9]

<u>Solution:</u>
The circumference of the circle is 2π, so P performs $\frac{12\pi}{2\pi} = 6$ revolutions per second and Q performs $\frac{5\pi}{2\pi} = \frac{5}{2}$ revolutions per second. So, after time t, the number of revolutions performed by P is $6t$ and the number of revolutions performed by Q is $\frac{5t}{2}$. So, they meet for the third time when $6t = \frac{5t}{2} + 3$, that is, when $t = \frac{6}{7}$. Then, the distance moved by P is $12\pi \cdot \frac{6}{7} = \frac{72\pi}{7} = 10\pi + \frac{2\pi}{7}$; that is, 5 revolutions and a further distance of $\frac{2\pi}{7}$. So, the length of the minor arc AP is then $\frac{2\pi}{7}$, and the answer to the question is 9.

CSAML 2021

April 7, 2021
Round III: Geometry

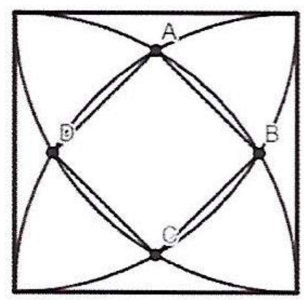
1. (1 point) If $m\angle A = (6x + 2y)°$ and $m\angle B = \{(15y)°, (3y)°, (6y)°, (5y)°\}$, angles A and B are congruent, and angles A and B are supplementary, find the value of x. [Answer: $13, 5, 10, 9$]

Solution:
Since the angles are congruent and supplementary, it follows that they are right angles. Setting $15y = 90$ yields $y = 6$, and therefore $6x + 2(6) = 90$, so $x = 13$.

2. (2 points) A rhombus has diagonals with lengths in the ratio of 2:1 and a perimeter of $\{10,30,40,50\}$. Find the area of the rhombus. [Answer: $5, 45, 80, 125$]

Solution:
Let the smaller diagonal have a length of $2n$ and the longer diagonal have a length of $4n$. Therefore, the side of the rhombus would have a length of $\sqrt{n^2 + 4n^2} = n\sqrt{5}$. The perimeter is therefore $4n\sqrt{5}$, so $n = \frac{10}{4\sqrt{5}}$. Since the area of a rhombus is $\frac{1}{2}d_1 d_2$, the area is $\frac{1}{2}\left(\frac{20}{4\sqrt{5}}\right)\left(\frac{40}{4\sqrt{5}}\right) = \frac{800}{160} = 5$.

3. (3 points) Four quarter circles of radius 1 are drawn in a unit square such that the center of each quarter circle is a vertex of the square as shown in the figure. The intersections of the quarter circles are connected to form square *ABCD*. The area of $ABCD$ can be expressed as $m - \sqrt{n}$ where m and n are positive integers. Find the value of $\{\frac{1}{2}m + 3n, 2m - n, m + 2n, 7m - 4n\}$. [Answer: $10, 1, 8, 2$]



Solution:
One way to solve this is to find the length of one diagonal d and compute the area as $\frac{1}{2}d^2$. Since point A is a distance of $\frac{\sqrt{3}}{2}$ from the bottom side of the square, it follows that AC is $1 - 2\left(1 - \frac{\sqrt{3}}{2}\right) = \sqrt{3} - 1$. Then the area is $\frac{1}{2}(\sqrt{3} - 1)^2 = \frac{1}{2}\left(4 - 2\sqrt{3}\right) = 2 - \sqrt{3}$. Therefore $m = 2$ and $n = 3$, so $\frac{1}{2}(2) + 3(3) = 10$.

CSAML 2021

April 7, 2021
Round IV: Algebra II

1. (1 point) Let $f(x) = x^3 + Ax^2 + Bx + C$, where A, B, C are constants. The graph of $y = f(x)$ is tangent to the *x*-axis at the point (1, 0) and passes through the point ({6,5,4,3}, 0). Find $|A| + |B| + |C|$.
[Answer: 27, 23, 19, 15]

<u>Solution:</u>
Since the graph of f is tangential to the *x*-axis at (1, 0) and passes through (6, 0), $f(x) = k(x-1)^2(x-6)$, and $k = 1$ since the leading coefficient in the given equation for $f(x)$ is 1. So $f(x) = (x-1)^2(x-6) = x^3 - 8x^2 + 13x - 6$. So, $|A| + |B| + |C| = 8 + 13 + 6 = 27$.

2. (2 points) Let $\log 2 = a$ and $\log 3 = b$.
Then $\{\log_{250} 18, \log_{250} 12, \log_{250} 24, \log_{250} 54\} = \frac{pa+qb}{r-sa}$,
where p, q, r and s are positive integers, and p and r are relatively prime. Find $p + 2q + 3r + 4s$.
(Here, $\log x$ is understood to mean $\log_{10} x$.)
[Answer: 22, 21, 22, 24]

<u>Solution:</u>

$$\log_{250} 18 = \frac{\log 18}{\log 250} = \frac{\log(2 \cdot 3^2)}{\log(10^3/2^2)} = \frac{\log 2 + 2\log 3}{3\log 10 - 2\log 2} = \frac{a + 2b}{3 - 2a}$$

So, $p + 2q + 3r + 4s = 1 + 2 \cdot 2 + 3 \cdot 3 + 4 \cdot 2 = 22$.

3. (3 points) Let $f(x) = \left\{\frac{1}{4}, \frac{1}{3}, \frac{1}{4}, \frac{1}{3}\right\} x$ and $g(x) = x^3 + x$. One of the points of intersection of the graphs of $y = f(x)$ and $y = g^{-1}(x)$ is (p, q), where $p > 0$ and $q > 0$. Find $\{2p^2 + q^2, 2p^2 + q^2, p^2 + 2q^2, p^2 + 2q^2\}$.
[Answer: 99, 38, 54, 22]

<u>Solution:</u>
It is very hard to determine the inverse of the function *g*; it's easier to find the intersection of $y = f^{-1}(x)$ and $y = g(x)$ and to reflect the point in the line $y = x$. $f^{-1}(x) = 4x$ and $g(x) = x + x^3$, so the *x*-coordinate of the intersection(s) is given by the equation $4x = x + x^3$, which has solutions $x = 0$ and $x = \pm\sqrt{3}$. It is given that the *y*-coordinate of the required point is positive, so we must select $x = \sqrt{3}$. Then $y = 4x = 4\sqrt{3}$. Reflecting in the line $y = x$ is equivalent to exchanging the coordinates, so the required intersection point is $(4\sqrt{3}, \sqrt{3})$, and $2p^2 + q^2 = 2 \cdot 48 + 3 = 99$.

CSAML 2021

April 7, 2021
Round V: Analytic Geometry

1. (1 point) A line in the coordinate plane has equal *x*- and *y*-intercepts and passes through the point $\{(-4,-8),(-2,-6),(-3,-7),(-5,-1)\}$. Determine the area of the right triangle formed by the line and the coordinate axes. [Answer: $72, 32, 50, 18$]

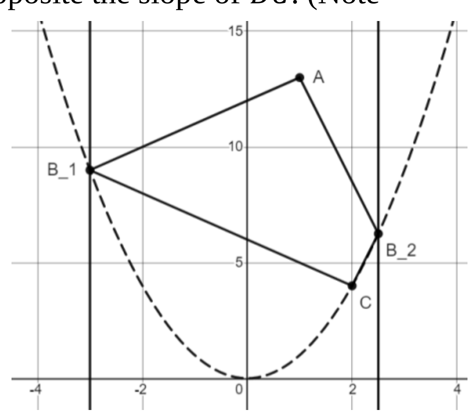
Solution:
Note that since the values of the intercepts are the same, the equation of the line must be in the form $x + y = C$. In this case, $C = -12$, so the legs of the triangle will each have length 12, making the area $\frac{1}{2}(12)^2 = 72$.

2. (2 points) A hyperbola has equation $x^2 - y^2 - 2ax + 4by + a^2 - 4b^2 - 1 = 0$, where a and b are constants. The asymptotes of the hyperbola have equations $\{x + y = 11, x + y = 10, x + y = 14, x + y = 11\}$ and $\{x - y = -1, x - y = 6, x - y = -2, x - y = 3\}$. Find $5a + 3b$.
[Answer: $34, 43, 42, 41$]

Solution:
We can rewrite the equation as $x^2 - 2ax + a^2 - (y^2 - 4by + 4b^2) = 1$, which is equivalent to $(x-a)^2 - (y-2b)^2 = 1$. Since the point $(a, 2b)$ must be the intersection of the asymptotes, we can solve the system from the equations for the asymptotes to get (5,6). This means $a = 5$ and $b = 3$, so $5a + 3b = 34$.

3. (3 points) Points *A*, *B*, and *C* have coordinates (1, {13,19,26,34}), (k, k^2), and $(2, 4)$ respectively. There are two possible values of k such that $4 < k^2 <$ {13,19,26,34} and the acute angle between $\overline{AB}$ and the line $x = k$ is congruent to the acute angle between $\overline{BC}$ and the line $x = k$. If the two possible values of k are k_1 and k_2, find the value of $|2k_1k_2|$. [Answer: $15, 21, 28, 36$]

Solution:
See the diagram. We are looking for point B such that the slope of $\overline{AB}$ is opposite the slope of $\overline{BC}$. (Note that k cannot be between 1 and 2 since the point would not lie on $y = x^2$.) Setting up the algebraic equation for this relationship gives $\frac{k^2-13}{k-1} = -\frac{k^2-4}{k-2} \rightarrow \frac{k^2-13}{k-1} = -(k+2) \rightarrow k^2 - 13 = -(k^2 + k - 2)$, which simplifies to $2k^2 + k - 15 = 0$. We could now solve for the two values of k, or we can use properties of quadratics to know from the equation that the product of the zeros must be $-\frac{15}{2}$, so $|2k_1k_2| = 15$.



CSAML 2021

April 7, 2021
Round VI: Trigonometry and Complex Numbers

1. (1 point) Let A be the point (4, 3), B be the point (2, {5,6,7,8}), and O be the point (0, 0). Find the measure of angle AOB, rounding your answer to the nearest degree. (Do not include a unit in your answer.)
[Answer: 31, 35, 37, 39]

Solution:
Let P be a point on the x-axis with positive x-coordinate. $m\angle AOB = m\angle BOP - m\angle AOP$
$= \tan^{-1}\frac{5}{2} - \tan^{-1}\frac{3}{4} = 31.3°$. Thus, the answer to the question is 31.

2. (2 points) $i + 2i^2 + 3i^3 + \cdots + \{1004,1008,1012,1016\}i^{\{1004,1008,1012,1016\}} = A + Bi$, where A and B are real.
Find $2|A| + |B|$.
[Answer: 1506, 1512, 1518, 1524]

Solution:
Note first that $i + 2i^2 + 3i^3 + 4i^4 = i - 2 - 3i + 4 = 2 - 2i$. Also, $5i^5 + 6i^6 + 7i^7 + 8i^8 = 5i - 6 + 7i + 8 = 2 - 2i$. Similarly, the sum of each set of four consecutive terms is $2 - 2i$. So, the sum of the entire series is $(1004/4)(2 - 2i) = 502 - 502i$. Thus, the answer to the question is $2 \cdot 502 + 502 = 1506$.

3. (3 points) Let $t = \tan x$. Then, for all values of x apart from odd integer multiples of $\frac{\pi}{6}$,

$$\frac{\sin 6x}{1 + \cos 6x} = \frac{At^3 + Bt^2 + Ct + D}{Et^2 + Ft + G}$$

where the coefficients $A, B, \ldots, G$ are integers, some of them zero, and A and E are relatively prime. Find $\{2|A| + |B| + |C| + |D| + |E| + |F| + |G|, 3|A| + |B| + |C| + |D| + |E| + |F| + |G|, 2A^2 + B^2 + C^2 + D^2 + E^2 + F^2 + G^2, 3A^2 + B^2 + C^2 + D^2 + E^2 + F^2 + G^2\}$.
[Answer: 9, 10, 21, 22]

Solution:
We use the formulas $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = 2\cos^2\theta - 1$.

$$\frac{\sin 6x}{1 + \cos 6x} = \frac{2\sin 3x \cos 3x}{1 + 2\cos^2 3x - 1} = \frac{\sin 3x}{\cos 3x} = \tan 3x = \tan(2x + x) = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$$

$$= \frac{\frac{2t}{1-t^2} + t}{1 - \frac{2t}{1-t^2} \cdot t} = \frac{2t + t(1-t^2)}{(1-t^2) - 2t^2} = \frac{3t - t^3}{1 - 3t^2} = \frac{t^3 - 3t}{3t^2 - 1}$$

So, $A = 1, C = -3, E = 3, G = -1$, and all the other coefficients are zero. Therefore, $2|A| + |B| + |C| + |D| + |E| + |F| + |G| = 2 + 3 + 3 + 1 = 9$.

CSAML 2021

April 7, 2021
Team Round

1. How many positive integer bases a such that $a \leq 2021$ satisfy the inequality $105_a > 10_{6a+2021}$?
[Answer: 1973]

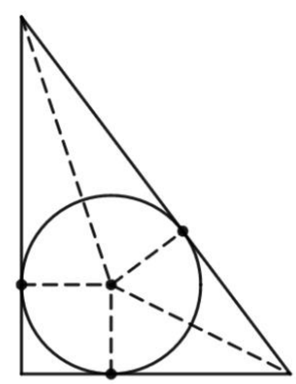
Solution:
Turning the base notation into an algebraic equation in base 10 yields $a^2 + 5 > 6a + 2021$, which can be written as $a^2 - 6a + 5 > 2021$. Adding four to both sides of the inequality completes the square, giving $(a-3)^2 > 2025$, and since a is positive, $a - 3 > 45$, so $a > 48$. This means all values of a from 49 to 2021 work, making 1973 values.

2. Find the area of the trapezoid with vertices (5, 6), (11,10), (16, 9), and (4, 1).
[Answer: 39]

Solution:
Let the given four points be *A*(5, 6), *B*(11,10), *C*(16, 9), and *D*(4, 1). Note that segments *AB* and *DC* are parallel, so they form the bases of the trapezoid. Next, notice that $AD = BC = \sqrt{26}$, telling us that the trapezoid is isosceles. Let the foot of the perpendicular from *A* to line *BC* be *P*. Then, using the symmetry of the trapezoid, $DP = (1/2)(DC - AB) = (1/2)\left(4\sqrt{13} - 2\sqrt{13}\right) = \sqrt{13}$. Applying the Pythagorean theorem to triangle *APD* we see that $AP = \sqrt{\left(\sqrt{26}\right)^2 - \left(\sqrt{13}\right)^2} = \sqrt{13}$, and this is the height of the trapezoid. So, the required area is $(1/2)\left(4\sqrt{13} + 2\sqrt{13}\right)\left(\sqrt{13}\right) = 39$.

3. A circle with an area of π is inscribed in a right triangle of area 12. Find the length of the hypotenuse of the right triangle.
[Answer: 11]

Solution:
See the diagram (not drawn to scale). The right triangle can be divided into a unit square and two pairs of congruent triangles. This means that if the legs of the triangle have lengths a and b, the hypotenuse has length $a - 1 + b - 1$. Knowing this gives us two equations in a and b: $\frac{1}{2}ab = 12$ and $a + b - 2 = \sqrt{a^2 + b^2}$. Squaring both sides of the second equation gives $a^2 + 2ab + b^2 - 4(a+b) + 4 = a^2 + b^2$, which can be simplified to $2ab = 4(a + b - 1)$, and dividing both sides by 4 gives $\frac{1}{2}ab = a + b - 1$. Knowing the area is twelve gives $12 = a + b - 1$, which means $12 - 1 = a + b - 2$, so the hypotenuse has length 11.

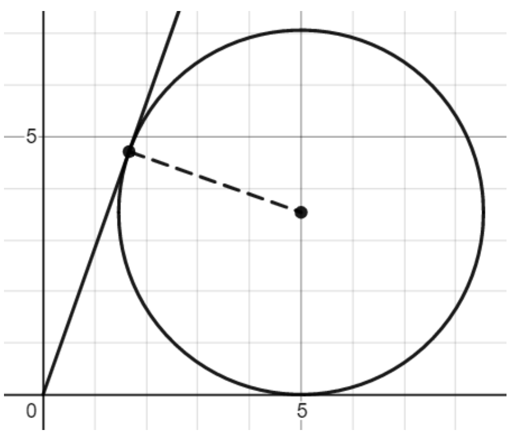


4. Let $f(x) = \frac{b}{9}x^2 - bx + 3b$, where b is a real constant. The remainder when $f(x)$ is divided by $x - 3b$ is 28. $f(2) = \frac{p}{q}$, where p and q are relatively prime integers and $q > 0$. Find $p + q$.
[Answer: 61]

Solution:
By the remainder theorem, $28 = f(3b) = \frac{b}{9}(3b)^2 - b(3b) + 3b$. From this we get that $b^3 - 3b^2 + 3b - 28 = 0$. This can be written as $b^3 - 3b^2 + 3b - 1 = 27$, so $(b-1)^3 = 27$. The only real solution to this equation is given by $b - 1 = 3$, so $b = 4$. Thus, $f(2) = (4/9) \cdot 2^2 - 4 \cdot 2 + 3 \cdot 4 = 52/9$. So, the answer to the question is $52 + 9 = 61$.

5. The circle $x^2 + y^2 - 10x - py + k = 0$ where $p > 0$ is tangent to both the x-axis and the line $y = 2\sqrt{2}x$. Find the value of kp^2.
[Answer: 1250]

Solution:



See the diagram. First, we know from the equation of the circle that the x-coordinate of the center is 5. This means the distance from the origin to either point of tangency is 5. Next, we can solve for the x-coordinate of the point of tangency along the line $y = 2\sqrt{2}x$ using $x^2 + (2\sqrt{2}x)^2 = 25x^2$, so $x = \frac{5}{3}$ and the point of tangency has coordinates $\left(\frac{5}{3}, \frac{10\sqrt{2}}{3}\right)$. One way to get the y-value of the center is to construct the equation of the line including the previous point of tangency and the center, $y = -\frac{1}{2\sqrt{2}}\left(x - \frac{5}{3}\right) + \frac{10\sqrt{2}}{3}$, and substituting in $x = 5$ gives a y −coordinate of $\frac{5\sqrt{2}}{2}$. This gives a full equation of the circle of $(x-5)^2 + (y - \frac{5\sqrt{2}}{2})^2 = \left(\frac{5\sqrt{2}}{2}\right)^2$, and expanding this gives $x^2 + y^2 - 10x - 5\sqrt{2}y + 25 = 0$, so $kp^2 = (25)(5\sqrt{2})^2 = 1250$.

6. Among complex numbers z that have the property that $|z - (8 + 6i)| = 3$, what is the maximum value of $|z|$?
[Answer: 13]

Solution:
The quantity $|z - (8 + 6i)|$ is the distance of z from the complex number $8 + 6i$. Therefore, the complex numbers z with $|z - (8 + 6i)| = 3$ form a circle with center $Q(8, 6)$ and radius 3. Letting the origin be O, the distance OQ is 10. Now consider points P on the circle. The distance OP is greatest when OQP is a straight line. Then, $|z| = OP = 10 + 3 = 13$.