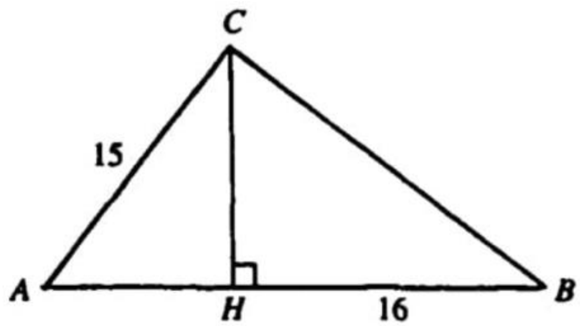
Connecticut ARML Team, 2021
Practice 1
May 15, 2021

Individual Questions

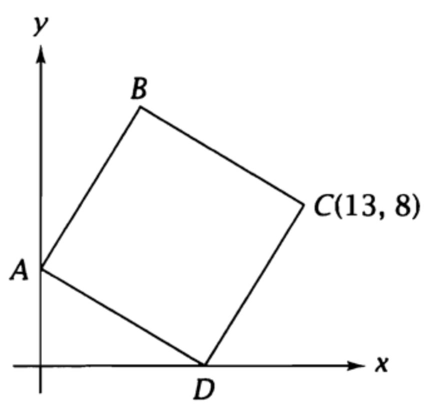
I-1. How many 4-digit numbers N are there such that some digit of N occurs exactly 3 times, and N is a multiple of 25? (Note: A 4-digit number cannot start with a zero.)

I-2. Right triangle ABC with hypotenuse AB has side $AC = 15$. Altitude CH divides AB into segments AH and HB, with $HB = 16$. Compute the area of triangle ABC.



I-3. An 18th-century American president held a party. Thirteen man-woman married couples attended the party. Each man shook hands with every other person at the party except his wife, and no handshakes took place between women. How many handshakes took place?

I-4. Square $ABCD$ is positioned on the axes as shown. Compute the area of the square.



I-5. There is only one value of x for which the median and the arithmetic mean of the five numbers 14, 12, 26, 16, and x are equal. Compute that value of x.

I-6. Consider a parabola whose axis is vertical and for which the vertex is the lowest point on the curve. The parabola passes through the origin and is tangent to the hyperbola $x^2 - 4y^2 = 25$. Compute the *y*-coordinate of the focus of the parabola.
[A fact you should know: The focus of the parabola $y^2 = 4ax$ is located at the point (*a*, 0).]

I-7. A quadrilateral is circumscribed about a circle. Three sides of the quadrilateral have lengths 17, 18, and 21, not necessarily in that order. Compute the smallest possible length of the fourth side.

I-8.

$$\left(\sin\frac{\pi}{16}\right)\left(\sin\frac{3\pi}{16}\right)\left(\sin\frac{5\pi}{16}\right)\left(\sin\frac{7\pi}{16}\right) = \frac{1}{2^{m/n}}.$$

where *m* and *n* are relatively prime positive integers. Compute $m + n$.

I-9. Suppose that x, y, z are 2-digit integers and $8^x + 16^y = 32^z$. Compute the largest possible value for $x + y + z$,

I-10. The smallest value of *x* such that $\lfloor 3x \rfloor + \lfloor 5x \rfloor = 84$ is $\frac{m}{n}$, where *m* and *n* are relatively prime positive integers. Compute $m + n$.
[You should know that $\lfloor x \rfloor$ is the greatest integer not exceeding *x*.]

Answers to Individual Questions

I-1. 14
I-2. 150
I-3. 234
I-4. 89
I-5. 2
I-6. 5
I-7. 14
I-8. 9
I-9. 207
I-10. 58

Solutions to Individual Questions

I-1. *N* must end in 00, 25, 50, or 75. If *N* ends in 00, the possibilities for *N* are 1000, 2000, …, 9000. If *N* ends in 25, the possibilities are 2225 and 5525. If *N* ends in 50, the only possibility is 5550. If *N* ends in 75, the possibilities are 7775 and 5575. There are, in total, 14 possibilities.

I-2. Let $AH = x$ and $HC = h$. By the similarity of triangles *CAB* and *HAC*, $15/(16+x) = x/15$. This tells us that $x^2 + 16x - 225 = 0$, for which the only positive solution is $x = 9$. By the similarity of triangles *CHB* and *AHC*, $h/16 = x/h$, so $h^2 = 16x = 16 \cdot 9$, so $h = 4 \cdot 3 = 12$. Thus, the area of triangle *ABC* is $(1/2)(x+16)h = (1/2) \cdot 25 \cdot 12 = 150$.

I-3. The number of handshakes between men is ${}_{13}C_2 = 78$. The number of man-woman handshakes is $13 \cdot 12 = 156$. So, the total number of handshakes is 78 + 156 = 234.

I-4. Drop a perpendicular $\overline{CP}$ from *C* to the *x*-axis; call the origin *O*. From congruent triangles *AOD* and *DPC*, *OD* = *PC* = 8, so *DP* = 5. By the Pythagorean Theorem, $CD^2 = 89$, which is the area of the square.

I-5. The mean is $(68 + x)/5$. If $x \geq 16$, median = 16, so $(68+x)/5 = 16$ produces $x = 2$, a contradiction. If $14 < x < 16$, median = *x*, so $(68+x)/5 = x$ produces $x = 17$, a contradiction. If $x \leq 14$, median = 14, which leads to $x = 2$.

I-6. By symmetry, the equation of the parabola must be $y = kx^2$, for some $k > 0$. By substituting for x^2 in the equation of the hyperbola we see that, at the points of intersection, $\frac{y}{k} - 4y^2 = 25$; that is, $4ky^2 - y + 25k = 0$. For the parabola to be tangential to the hyperbola, this equation must have exactly one solution, so its discriminant must be zero. This tells us that $1 - 16 \cdot 25k^2 = 0$, so $k = 1/20$. Thus, the equation of the parabola is $x^2 = 20y$, and so the *y*-coordinate of its focus is 5.

I-7. For any quadrilateral *ABCD* circumscribed about a circle, let the points of tangency of segments *AB*, *BC*, *CD*, and *DA* be *P*, *Q*, *R*, and *S*, respectively. Let *AS* = *AP* = *w*, *BP* = *BQ* = *x*, *CQ* = *CR* = *y*, and *DR* = *DS* = *z*. Then $AB + CD = w + x + y + z = BC + DA$. We have thus established the fact that, when a quadrilateral is circumscribed about a circle, the sum of the lengths of one pair of opposite sides must equal the sum of the lengths of the other pair. In the context of this problem, for the fourth side to be as small as possible, let two opposite sides have lengths 17 and 18. The fourth side then has length 14.

I-8. Note, first, that $\left(\sin\frac{\pi}{16}\right)\left(\sin\frac{7\pi}{16}\right) = \left(\sin\frac{\pi}{16}\right)\left(\cos\frac{\pi}{16}\right) = \frac{1}{2}\sin\frac{\pi}{8}$. Similarly, $\left(\sin\frac{3\pi}{16}\right)\left(\sin\frac{5\pi}{16}\right) = \frac{1}{2}\sin\frac{3\pi}{8}$. Thus, the required quantity is $\frac{1}{2^2}\sin\frac{\pi}{8}\sin\frac{3\pi}{8} = \frac{1}{2^2}\sin\frac{\pi}{8}\cos\frac{\pi}{8} = \frac{1}{2^3}\sin\frac{\pi}{4} = \frac{1}{2^{7/2}}$. So, the answer to the question is $7 + 2 = 9$.

I-9. From $8^x + 16^y = 32^z$ we see that $2^{3x} + 2^{4y} = 2^{5z}$. The left-hand side of this equation is thus a power of 2, and for this to be true we must have $3x = 4y$. Thus, *x* is a two-digit number divisible by 4. Furthermore, we have that $2^{3x} + 2^{3x} = 2^{5z}$, so $2^{3x+1} = 2^{5z}$. So, $3x + 1 = 5z$, telling us that $3x \equiv -1 \pmod 5$. From this we get that $x \equiv 3 \pmod 5$. To summarize, *x* is a two-digit number divisible by 4 and congruent to 3 (mod 5). The largest such number is 88. (Note that by making x as large as possible we make y and z as large as possible, and so make $x + y + z$ as large as possible.) So $x = 88$, $y = 3x/4 = 66$, and $z = (3x + 1)/5 = 53$. Thus, $x + y + z = 207$.

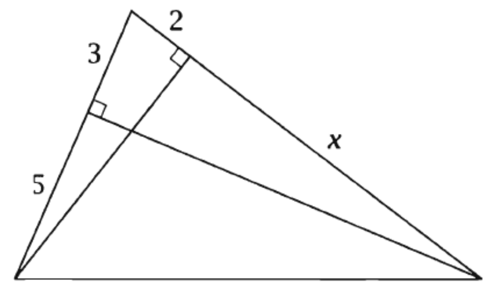
I-10. Let $f(x) = \lfloor 3x \rfloor + \lfloor 5x \rfloor$. Note that $f(10) = 80$ and $f(11) = 88$, so the required value of x is between 10 and 11. As x increases from 10, every 1/3 of a unit increase in x results in an increase of 1 in $\lfloor 3x \rfloor$. Also, every increase of 1/5 in x results in an increase of 1 in $\lfloor 5x \rfloor$. We conclude from this that f takes the table of values shown below, where each value of x given in the table is the smallest that results in the given value of y.

x	$f(x)$
10	80
10.2	81
$10\frac{1}{3}$	82
10.4	83
10.6	84
$10\frac{2}{3}$	85
11	86

The required value of x is therefore $10.6 = \frac{53}{5}$, and the answer to the question is 58.

Team Round 1 (Teams of 5; 15 minutes)

T1. Two of the altitudes of an acute triangle cut the sides to which they are drawn into segments of lengths 5, 3, 2, and x, as shown. Compute x.



T2. A line with slope -2 is 2 units from the origin. Compute the area of the triangle formed by this line and the coordinate axes.

T3. Compute the smallest positive integer that exactly divides 16! but does *not* divide 14! .

T4. Compute the number of ordered pairs (x, y) with the property that x and y are each 2-digit numbers, $x < y$, and $x \cdot y$ is a 3-digit number whose three digits are identical.

T5. The number $(9^6 + 1)$ is the product of three primes. Compute the largest of these primes.

T6. There are two 4-digit numbers, ABCA, with the property that AB is a prime, BC is a square, and CA is the product of a prime and a square that is greater than 1. Compute the larger of these 4-digit numbers.

T7. If $f(x) = |3x - 1|$, compute all values of x for which $f(f(x)) = x$.

Team Round 1 Solutions (NYSML 1992)

T1. Label the lower left vertex A, upper vertex B, other vertex C, altitudes $\overline{AE}$ and $\overline{CD}$.

Method 1: *SIMILARITY.* From similar triangles ABE and CBD, $2/8 = 3/(2+x)$, so $x = 10$. [If you prefer a *TRIGONOMETRIC* approach, each of these fractions is $\cos B$.]

Method 2: *CIRCLES.* A circle with diameter $\overline{AC}$ must pass through D and E. Using a standard theorem about secants [$\overline{BDA}$ and $\overline{BEC}$] to a circle from an outside point, we have $3(8) = 2(2+x)$, so $x = 10$.

Method 3: *PYTHAGOREAN THEOREM.* In right triangle ABE, $AE = \sqrt{60}$; thus in right triangle ACE, $AC = \sqrt{60+x^2}$; then in right triangle ACD, $CD = \sqrt{35+x^2}$; finally from right triangle BCD, $3^2 + (35+x^2) = (2+x)^2$, so $x = 10$.

Method 4: *AREA.* From right triangles ABE and BCD we get $AE = \sqrt{60}$ and $CD = \sqrt{x^2+4x-5}$. Using these altitudes and the corresponding bases to get the area of triangle ABC leads to $(2+x)\sqrt{60} = 8\sqrt{x^2+4x-5}$, so $x^2+4x-140 = 0$, producing $x = 10$ as the only positive root.

This simple problem is certainly rich in mathematics.

T2. Let the legs of the triangle be a and $2a$; then the hypotenuse is $a\sqrt{5}$ (and the altitude to the hypotenuse is given as 2). The area $K = (1/2)(2a)(a) = (1/2)(2)(a\sqrt{5})$, so $a = \sqrt{5}$ and $K = 5$.

T3. Factoring, we find that $14! = 2^{11} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$ and that $16! = 2^{15} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$. Thus our answer is the smallest of the numbers $2^{12}, 3^6$, and 5^3. The answer is 125 [or 5^3].

T4. Since $x \cdot y = \text{DDD} = D(111) = D(3 \cdot 37)$, we can have $(x,y) = (3D, 37)$ provided that $D \geq 4$ OR $(x,y) = (3 \cdot \frac{D}{2}, 74)$ if D is 8. There are seven such pairs.

T5. Using the fact that $x^3+1 = (x+1)(x^2-x+1)$, we have $9^6+1 = (9^2)^3+1 = (9^2+1)(9^4-9^2+1) = 82 \cdot 6481 = 2 \cdot 41 \cdot 6481$. From the given information, it must be true that 6,481 is a prime, and that must be the correct answer.

T6. Since AB is a prime, B must be 1,3,7, or 9. There are no squares in the 70's or 90's, so B = 1 or 3 and C = 6. A quick check of the factors of the integers from 61 through 69 shows that only 63 and 68 satisfy the third condition of the problem, so A = 3 or 8. Remembering condition one, the only 4-digit numbers that work are 3,163 and 8,368.

T7. $f(f(x)) = |\,3|3x-1|-1\,| = x$ implies $3|3x-1|-1 = \pm x$, so $3|3x-1| = 1 \pm x$. If $x \geq 1/3$, we have $9x - 3 = 1 \pm x$, leading to $x = 1/2$ or $2/5$, both of which check. If $x < 1/3$, we have $3 - 9x = 1 \pm x$, leading to $x = 1/5$ or $1/4$. Answer: 1/4, 1/5, 2/5, 1/2.

Relay 1

R1-1. At Ridgemont High School, 10% of the female students and 5% of the male students are on the math team. 80% of all the students in the school are male. If a member of the math team is selected at random, with each member of the team having an equal chance of being selected, compute the probability that the selected member is male.

R1-2. Let t be TNYWR, and let $k = 15t$. Compute the sum of the infinite series $\frac{k}{2} + \frac{k}{4} - \frac{k}{8} - \frac{k}{16} + \frac{k}{32} + \frac{k}{62} -$ ... (where each denominator is twice the previous one, and the pattern of 2 pluses, then two minuses, then 2 pluses, etc., continues).

R1-3. Let t be TNYWR (which will be a positive integer). If $a^{\frac{1}{2}} + b^{\frac{1}{2}} = t$, compute the *number* of ordered pairs of *positive integers* (a, b) that satisfy the equation.

Relay 1 Solutions (ARML 1986, Relay 2, Q1–3)

R1-1. Rather than solve this generally, it is easiest to let there be 200 students in the school, so 160 are male and 40 are female. That means the math team consists of 8 males and 4 females. The required probability is 8/12, which is 2/3.

R1-2. $t = 2/3$ so $k = 10$. Factoring out a k and combining pairs of terms, the series becomes $k\left[\frac{3}{4} - \frac{3}{16} + \frac{3}{64} - \cdots\right] = k\left[\frac{3}{5}\right]$ by using the formula for the sum of an infinite geometric series. The answer is 6.

R1-3. $t = 6$. If the sum of two positive integers is t, then the first can be any integer from 1 through $t - 1$. This gives $t - 1$ ordered pairs, so the answer is 5. For example, $4 + 2 = 6$, so a can be 16 and b would be 4; $a^{1/2}$ ranges from 1 through 5.

Relay 2

R2-1. Compute the number of ordered pairs (x, y) of positive integers satisfying $x^2 - 8x + y^2 + 4y = 5$.

R2-2. Let $T = TNYWR$ and let $k = 21 + 2T$. Compute the largest integer n such that $2n^2 - kn + 77$ is a positive prime number.

R2-3. Let $T = TNYWR$. In triangle ABC, $BC = T$ and $m\angle B = 30°$. Compute the number of integer values of AC for which there are two possible values for side length AB.

Relay 2 Solutions (ARML Local 2009, Relay 2)

R2-1. Completing the square twice in x and y, we obtain the equivalent equation $(x-4)^2 + (y+2)^2 = 25$, which describes a circle centered at $(4, -2)$ with radius 5. The lattice points on this circle are points 5 units up, down, left, or right of the center, or points 3 units away on one axis and 4 units away on the other. Because the center is below the x-axis, we know that y must increase by at least 2 units; x cannot decrease by 4 or more units if it is to remain positive. Thus, we have:

$$(x, y) = (4, -2) + (-3,4) = (1,2)$$
$$(x, y) = (4, -2) + (0,5) = (4,3)$$
$$(x, y) = (4, -2) + (3,4) = (7,2)$$
$$(x, y) = (4, -2) + (4,3) = (8,1)$$

There are **4** such ordered pairs.

R2-2. If k is positive, there are only four possible factorizations of $2n^2 - kn + 77$ over the integers, namely

$$(2n-77)(n-1) = 2n^2 - 79n + 77$$
$$(2n-1)(n-77) = 2n^2 - 145n + 77$$
$$(2n-11)(n-7) = 2n^2 - 25n + 77$$
$$(2n-7)(n-11) = 2n^2 - 29n + 77$$

Because $T = 4$, $k = 29$, and so the last factorization is correct one. Because $2n - 7$ and $n - 11$ are both integers, in order for their product to be prime, one factor must equal 1 or -1, so $n = 3, 4, 10$ or 12. Checking these possibilities from the greatest downward, $n = 12$ produces $17 \cdot 1 = 17$, which is prime. So the answer is **12**.

R2-3. By the Law of Cosines, $(AC)^2 = T^2 + (AB)^2 - 2T(AB)\cos 30° \rightarrow (AB)^2 - 2T\cos 30°(AB) + (T^2 - (AC)^2) = 0$. This quadratic in ABhas two positive solutions when the discriminant and product of the roots are both positive. Thus $(2T\cos 30°)^2 - 4(T^2 - (AC)^2) > 0$ and $(T^2 - (AC)^2) > 0$. The second inequality implies that $AC < T$. The first inequality simplifies to $4(AC)^2 - T^2 > 0$, so $\frac{T}{2} < AC$. Since $T = 12$, we have that $6 < AC < 12$, giving **5** integral values for AC.