Connecticut ARML Team, 2021
Practice 2
May 22, 2021

Individual Questions

I-1. In the sequence $x_1, x_2, x_3, \ldots,$ the first term, x_1, is 1 and, for $n > 1$, x_n is the smallest integer for which $x_n \cdot x_{n-1} \geq n^2 + n$. Compute x_{2021}.

I-2. Compute the number of triples of positive digits (A, B, C) such that $\underline{A}\,\underline{B} \cdot \underline{A}\,\underline{C} = \underline{B}\,\underline{A} \cdot \underline{C}\,\underline{A}$.
(Note: In ARML, underlined letters signify digits.)

I-3. In a triangle ABC, points K and M lie on $\overline{AC}$ and $\overline{BC}$ respectively such that $\overline{KM} \parallel \overline{AB}$. If L is an arbitrary point on $\overline{AB}$, the area of a triangle ABC is 9, and the area of a triangle KMC is 4, find the area of a quadrilateral $KLMC$.

I-4. Find the sum of all values of x that satisfy the inequality below.

$$\left(\sqrt{x^2 - 4x + 3} + 1\right)(\log_5 x - 1) + \frac{1}{x}\left(\sqrt{8x - 2x^2 - 6} + 1\right) \leq 0$$

I-5. Let $N = \underline{a}\,\underline{b}\,\underline{c}$ be a three-digit number, where the three digits are distinct and nonzero. Permuting the digits, five more three-digit numbers are created, and the sum of all six numbers is three times the number $\underline{a}\,\underline{a}\,\underline{a}$. Find the smallest such number N.

I-6. A fair die is rolled 3 times. The probability that the numbers rolled are rolled in a strictly increasing order is $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

I-7. There is one integer solution (a, b) to the equation $7a + 14b = 5a^2 + 5ab + 5b^2$ with $a > 0$. Compute the value of b.

I-8. Compute the sum of the digits of the following:

$$\sqrt{\underbrace{111\ldots1}_{2000} - \underbrace{222\ldots2}_{1000}}$$

(Note: $\underbrace{111\ldots1}_{2000}$ is the 2000-digit number 111… 1.)

I-9. Let $ABCD$ be a convex quadrilateral, and let diagonals $\overline{AC}$ and $\overline{BD}$ intersect at a point O. If $AB = 2CD$, $m\angle ADB = 90°$, and $m\angle ACB = 90°$, compute the degree measure of $\angle AOD$.

I-10. For a positive integer n, let $d(n)$ be the number of divisors of n (including 1 and n). Find the integer n closest to 1500 for which $d(n) + d(5n) = 15$.

Answers to Individual Questions

I-1. 1011
I-2. 17
I-3. 6
I-4. 1
I-5. 612
I-6. 59
I-7. 2
I-8. 3000
I-9. 60
I-10. 1445

Solutions to Individual Questions

I-1. *Answer:* 1011

Solution: By induction it can be shown that $x_n = \frac{n+1}{2}$ if n is odd and $x_n = 2n + 2$ if n is even. As a result, we have $x_{2021} = 1011$.

I-2. *Answer:* 17

Solution: Expanding and rearranging $(10A + B)(10A + C) = (10C + A)(10B + A)$ gives $99A^2 = 99BC$, or $A^2 = BC$. For $A = 1, 5, 7, 8, 9$, the only ordered triples that work are for $A = B = C$. For $A = p, p \leq 3$, we have 3 triples each, for $B = 1, p, p^2$. For $A = 4$ and 6 we have 3 triples each (for $B = \{2,4,8\}$ and $\{4,6,9\}$). Thus, there are **17** triples.

I-3. *Answer:* 6

Solution: Let $KM = x \cdot AB$. Then for altitudes, h and h_1 from point C, of triangles ΔABC and ΔKMC respectively we have: $h_1 = x \cdot h$. Since the area of $\Delta ABC = 9 = \frac{1}{2}AB \cdot h$ and the area of $\Delta KMC = 4 = \frac{1}{2}KM \cdot h_1$, we get that $4 = \frac{1}{2}x^2 AB \cdot h$, or that $4 = 9x^2$, therefore $x = \frac{2}{3}$. Since the altitude of ΔKLM is $h - h_1$, the area of quadrilateral $KLMC$ is:

$$A_{KLMC} = A_{\Delta KMC} + A_{\Delta KLM} = \frac{1}{2}KM \cdot h_1 + \frac{1}{2}KM(h - h_1) = \frac{1}{2}x \cdot AB \cdot h = \frac{2}{3} \cdot 9 = \mathbf{6}$$

I-4. *Answer:* 1

Solution: From the given we have that: $x > 0, x^2 - 4x + 3 \geq 0, -2x^2 + 8x - 6 \geq 0$. From the latter two inequalities we get that $x^2 - 4x + 3 = 0$, from where $x = 3$ or $x = 1$.

For $x = 1$, the inequality reduces to $\log_5 1 \leq 0$ and that is satisfied.

For $x = 3$, the inequality reduces to $\log_5 3 - \frac{2}{3} \leq 0$, or $3 \leq 5^{\frac{2}{3}}$, or $3^3 < 5^2$, which is not satisfied.

Therefore $x = \mathbf{1}$ is the only solution.

I-5. *Answer:* 612

Solution: Let $N = 100a + 10b + c$. In the given 6 permutations, every one of these digits (a, b, c) shows up twice as the "ones" digit, twice as the "tens" digit, and twice as the "hundreds" digit. Therefore, the sum of these six numbers is $222(c + b + a)$. Therefore, $222(c + b + a) = 3a \cdot 111$, so $a = 2(c + b)$. Since all the digits are different, we get that $a = 6$ or $a = 8$, so N is either 612, 621, 813, 831. The smallest is **612**.

I-6. *Answer:* 59

Solution: The odds of getting three different values are $\frac{6}{6} \cdot \frac{5}{6} \cdot \frac{4}{6} = \frac{20}{36} = \frac{5}{9}$. Given the event of rolling three different values, each of the $3! = 6$ orderings is equally likely, so the probability of getting an increasing ordering is $\frac{5}{9} \cdot \frac{1}{6} = \frac{5}{54}$. Therefore, the answer is $5 + 54 = \mathbf{59}$.

I-7. *Answer:* 2

Solution: If the given equation is written as a quadratic equation in terms of b we get the following:

$$5b^2 + (5a - 14)b + (5a^2 - 7a) = 0$$

Its discriminant is $196 - 75a^2$, which means that the equation has solutions for $a^2 \leq \frac{196}{75} = 2.613\ldots$. Therefore $a = -1, 0$ or 1. Since $a > 0$ is given, $a = 1$. Therefore $\boldsymbol{b = 2}$.

I-8. *Answer:* 3000

Solution: We have $\underbrace{111\ldots1}_{2000} - \underbrace{222\ldots2}_{1000} = \frac{1}{9}\left(\underbrace{999\ldots9}_{2000} - 2 \cdot \underbrace{999\ldots9}_{1000}\right) = \frac{1}{9}\left(10^{2000} - 1 - 2(10^{1000} - 1)\right) = \frac{1}{3^2}(10^{2000} - 2 \cdot 10^{1000} + 1) = \left[\frac{1}{3}(10^{1000} - 1)\right]^2 = \left(\frac{1}{3} \cdot \underbrace{999\ldots9}_{1000}\right)^2 = \underbrace{333\ldots3}_{1000}{}^2$. Thus, the value of the given expression is $\underbrace{333\ldots3}_{1000}$, of which the sum of the digits is 3000.

I-9. *Answer:* 60

Solution: Since $\angle ADB = \angle ACB = 90°$, we can conclude that points A, B, C, D lie on a circle, whose diameter is AB. Let M be the center of that circle. We then have $DC = \frac{AB}{2} = MD = MC$, so ΔDMC is equilateral. Since $\angle DMC = 60°$, we get that inscribed angle $\angle DBC = 30°$. From the ΔBOC we get that $\angle BOC = 60°$. $\angle AOD$ and $\angle BOC$ are vertical angles, therefore $\angle AOD = 60°$.

I-10. *Answer:* 1445

Solution: First note that the number of prime factors of *n* is at most 2, for if *n* had at least three prime factors then $d(n)$ would be at least $2 \cdot 2 \cdot 2 = 8$, and $d(5n)$ would be more than 8, making it impossible that $d(n) + d(5n) = 15$. So, we can write $n = p_1^{a_1} p_2^{a_2}$, where p_1 and p_2 are distinct prime numbers and $a_1 \geq 0$, $a_2 \geq 0$.

Suppose that *n* is not divisible by 5. Then $n = p_1^{a_1} p_2^{a_2}$, where neither p_1 nor p_2 is 5, and $5n = 5p_1^{a_1} p_2^{a_2}$, so $d(5n) = 2d(n)$. So, we need $15 = d(n) + d(5n) = 3d(n)$, so $d(n) = 5$. Thus, $n = p^4$, for some prime number $p \neq 5$. Note that $3^4 = 81$ and $7^4 = 2401$; the closer of these two numbers to 1500 is 2401.

Now suppose that *n* is divisible by 5. Then $n = 5^{a_1} p_2^{a_2}$ and $5n = 5^{a_1+1} p_2^{a_2}$, where $a_1 \geq 1$ and $a_2 \geq 0$. So, we need $15 = d(n) + d(5n) = (a_1 + 1)(a_2 + 1) + (a_1 + 2)(a_2 + 1)$ $= (2a_1 + 3)(a_2 + 1)$. Now, since $a_1 \geq 1$, $2a_1 + 3$ can't be 1 or 3, so either $2a_1 + 3 = 5$ and $a_2 + 1 = 3$, or $2a_1 + 3 = 15$ and $a_2 + 1 = 1$. That is, either $a_1 = 1$ and $a_2 = 2$, or $a_1 = 6$ and $a_2 = 0$. In the second case, $n = 5^6$, which is much greater than 1500. In the first case, $n = 5p_2^2$, and for *n* to be close to 1500 we need p_2^2 to be close to 300. Note that $17^2 = 289$ and $19^2 = 361$, of which 289 is the closer to 300, so the value of *n* taking this form that is closest to 1500 is $5 \cdot 17^2 = 1445$. This number is closer to 1500 than 2401 is, so the required value of *n* is 1445.

Power Round (60 minutes)

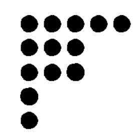
The numbers in square brackets represent the point value for each item.

Partitions

By a partition of a positive integer n, we mean a nonincreasing sequence of positive integers $(n_1, n_2, + \dots, n_k)$ such that $n_1 + n_2 + \dots + n_k = n$. The number of partitions of n is denoted by $p(n)$, so for example $p(5) = 7$ since it has the partitions (5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1). The number of integers occurring in a partition is called the number of parts. Throughout this problem n and k will refer to positive integers.

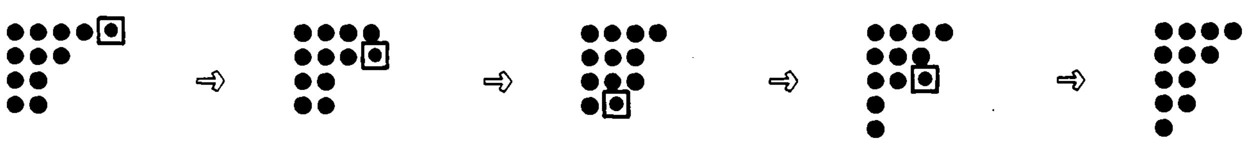
Ferrers Diagrams

A **diagram** of a partition, called a Ferrers Diagram, is a representation in which each part of the partition is represented by a row of dots, and the rows are placed under each other. For example the diagram for the partition (5, 3, 3, 1, 1) is show at right.



Stable Partitions

A **stabilizing move** on a diagram is defined as follows. (1) Find the highest row which has no dot one space down and to the left, then (2) move the last dot in this row to the end of the row below in, starting a new row if necessary. If there is no possible stabilizing move the partition is called **stable**. The number of stable partitions of n will he denoted by $s(n)$. The diagrams for the partition (5, 3, 2, 2) and its sequence of stabilizing moves are shown below. In each diagram the circled dot is the one that will be moved. The last partition is stable.



1. (a) Show, by a listing of diagrams, that $p(6) = 11$. [4]

   (b) Compute $s(6)$ by circling all the stable diagrams from part (a). [2]

2. Compute $s(9)$. [3]

3. Let $s(n, k)$ be the number of stable partitions of n whose largest part is k. For a given k find, with proof, an expression for the smallest value of n for which $s(n, k) > 0$. [3]

For items 4 to 7 we consider stabilizing moves starting with the partition (n). The sequence of partitions formed by making in succession all possible stabilizing moves will be called the **sequence of n,** and each partition in the sequence will be called a **stage**. For example the sequence of 4 is (4), (3, 1), (2, 2), (2, 1, 1) with 4 stages.

4. List the partitions or draw the diagrams for the sequence of 7. [3]

5. At certain stages in the sequence of n a new row is opened. The first time this happens is at stage 2. Prove that the next time it happens is at stage $[n/2]+2$, where $[x]$ is the greatest integer in x. [3]

6. Prove that for any n there will never be a stage in the sequence of n in which there are three rows of equal length, that is, one will never find a stage of the form $(\dots, k, k, k, \dots)$, [3]

7. For each integer n find. with proof, an expression for the stage, for n sufficiently large, in which the fourth row is opened. You will need to consider 3 cases. [4]

8. Let $f(n)$ be a function defined on the positive integers. The **generating function** for $f(n)$ is defined as the function $F(x)$ whose formal power series is $F(x)=\sum_{n=1}^{\infty} f(n)x^n$.

The power series is called formal because we don't worry about questions of convergence or summability For example the number of ways of obtaining n cents using only pennies, nickels and dimes is given by the coefficient of x^n in the expansion of

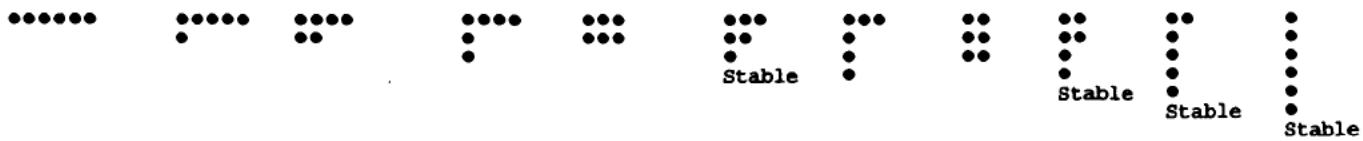
$$\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1}{1-x^{10}}\right) = (1+x+x^2+\ldots)(1+x^5+x^{10}+\ldots)(1+x^{10}+x^{20}+\ldots).$$

To see this consider the number of ways of obtaining x^{10} in the expansion. Each term will be the product of exactly one term from each of the geometric series in parentheses. We list the ways the result can be x^{10}, which will be the coefficient of x^{10} in the expansion: $(x^{10})(1)(1)$, $(x^5)(x^5)(1)$, $(1)(x^{10})(1)$, and $(1)(1)(x^{10})$. These correspond, respectively, to: (10, 0, 0), (5, 1, 0), (0, 2, 0), and (0, 0, 1) where the ordered triples list the number of pennies, nickels, and dimes used. This lists all the ways to obtain 10 cents using pennies, nickels and dimes. Thus there is a one to one correspondence between obtaining 10 cents and the coefficient of x^{10} in the expansion. This will be true for other values of n also.

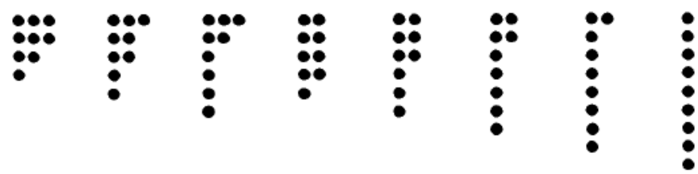
Find $S_k(x)$, the generating function for $s(n, k)$ as defined in item 3. [5]

Power Round Solutions (NYSML 2001)

1.



2. As the diagrams show $s(9) = 8$.



3. For a partition to be stable, as one moves down the diagram the length of rows can drop by at most one. In other words, if a row of length k exists, there must be at least one row each of lengths k, k-1, ... 2, 1. To use up the fewest number of dots, there must be exactly one row of each length.

Thus the smallest such n is $1+2+...+k=\dfrac{k(k+1)}{2}$

4. The partitions are (7), (6, 1), (5,2), (4,3), (4,2,1), (3,3,1), (3,2,2), (3,2,1,1)

5. Since a stabilizing move is made from the highest possible row, as moves are made the top row becomes smaller while the second row becomes larger until they are as close in length as possible. After this stage is reached the third row will open. If n is even this happens at the partition $(n/2, n/2) = ([n/2], [n/2])$, and if n is odd this happens at the partition $([n/2]+1, [n/2])$. In either case the second row has gone from length 0 to length $[n/2]$, so the partition is reached at stage $[n/2] + 1$. The third row will therefore open up at stage $[n/2] + 2$.

6. The proof is found by considering the stage immediately preceding the stage $(..., k, k, k, ...)$. If the first k was $k-1$ then the preceding stage was of the form $(..., k+m, k-1, k, k ...)$, with $m \geq 1$. If the second k was $k-1$ then the preceding stage was of the form $(..., k+1, k-1, k....)$. If the third k was $k-1$ then the preceding stage was of the form $(..., k, k+1, k-1, ...)$. In each of these cases the immediately preceding stage violates the nonincreasing condition for partitions and is thus impossible.

7. The fourth row will not open up until the first three rows are as close in length to each other as possible. The way this happens depends on the remainder n leaves when divided by three. The possibilities are

Case 1: $n = 3k$, with partition $(k+1, k, k-1)$, since (k, k, k) is impossible.
Case 2: $n = 3k+1$, with partition $(k+1, k, k)$.
Case 3: $n = 3k+2$, with partition $(k+1, k+1, k)$.

In general, to reach the stage (a, b, c) from the stage (n) requires $b + 2c$, moves since every dot in the second row has moved once and every dot in the third row has moved twice. Thus the number of moves required in each of the cases above is:

Case 1: $k + 2(k-1) = 3k-2 = n-2$. This occurs at stage $n-1$.
Case 2: $k + 2k = 3k = n-1$. This occurs at stage n.
Case 3: $k + 1 + 2k = 3k + 1 = n-1$. This also occurs at stage n.

Therefore the fourth row is opened at stage n if n is divisible by three, and at stage $n + 1$ otherwise.

8. We may think of partitions of n with parts at most size k as obtaining n cents using coins of value 1 cent, 2 cents, ..., k cents. The generating function for these partitions would be

$$\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\cdots\left(\frac{1}{1-x^k}\right)=\prod_{j=1}^{k}\frac{1}{1-x^j}=(1+x+x^2+\ldots)(1+x^2+x^4+\ldots)\ldots(1+x^k+x^{2k}+\ldots)$$

The symbol $\prod$ plays the same role for products as $\sum$ does for sums.

and this is the generating function for partitions with parts at most size k. (This also means that the generating function for $p(n)$ is

$P(x)=\prod_{k=1}^{\infty}\frac{1}{1-x^k}$, which is known as Euler's Identity.) However, not all of these partitions will be stable. In order to be stable, a partition must contain at least one row of each of the lengths 1, 2, ... , k. Since the 1 $(=x^0=(x^2)^0=\ldots)$) in each of the geometric series corresponds to choosing no row of that size, the desired generating function, $S_k(x)$, is

$$(x+x^2+\ldots)(x^2+x^4+\ldots)\ldots(x^k+x^{2k}+\ldots)=\left(\frac{x}{1-x}\right)\left(\frac{x^2}{1-x^2}\right)\cdots\left(\frac{x^k}{1-x^k}\right)=\prod_{j=1}^{k}\frac{x^j}{1-x^j}.$$

By gathering the powers of x in the numerator together, this can also be expressed as

$$s_k(x)=(x)(x^2)\ldots(x^k)\prod_{j=1}^{k}\frac{1}{1-x^j}=x^{1+2+\ldots+k}\prod_{j=1}^{k}\frac{1}{1-x^j}=x^{k(k+1)/2}\prod_{j=1}^{k}\frac{1}{1=x^j}.$$

which shows, as in item 3, that the smallest term with coefficient greater than 0 is the $x^{k(k+1+/2}$ term.

To find the generating function $S(x)$ for $s(n)$ just note that n could have stable partitions with largest part of size 1 or 2 or ... , which means that $S(x)=S_1(x)+S_2(x)+\ldots$, or

$$S(x)=\prod_{k=1}^{\infty}S_k(x)=\sum_{k=1}^{k}\prod_{j=1}^{k}\frac{x^j}{1-x^j}.$$

Using Mathematica to expand $S(x)$ gives the first 20 terms as

$$S(x)=x+x^2+2x^3+2x^4+3x^5+4x^6+5x^7+6x^8+8x^9+10x^{10}+12x^{11}+15x^{12}+18x^{13}+22x^{14}+27x^{15}+32x^{17}+46x^{19}+ 54x^{19}+64x^{20}+\ldots,$$

Team Round

T-1. Compute all ordered pairs of real numbers (x, y) that satisfy both of the equations:

$$x^2 + y^2 = 6y - 4x + 12 \qquad \text{and} \qquad 4y = x^2 + 4x + 12.$$

T-2. Define $\log^*(n)$ to be the smallest number of times the log function must be iteratively applied to n to get a result less than or equal to 1. For example, $\log^*(1000) = 2$ since $\log 1000 = 3$ and $\log(\log 1000) = \log 3 = 0.477\ldots \leq 1$. Let a be the smallest integer such that $\log^*(a) = 3$. Compute the number of zeros in the base 10 representation of a.

T-3. An integer N is worth 1 point for each pair of digits it contains that forms a prime in its original order. For example, 6733 is worth 3 points (for 67, 73, and 73 again), and 20304 is worth 2 points (for 23 and 03). Compute the smallest positive integer that is worth exactly 11 points. [Note: Leading zeros are <u>not</u> allowed in the original integer.]

T-4. The six sides of convex hexagon $A_1A_2A_3A_4A_5A_6$ are colored red. Each of the diagonals of the hexagon is colored either red or blue. Compute the number of colorings such that every triangle $A_iA_jA_k$ has at least one red side.

T-5. Compute the smallest positive integer n such that n^n has at least 1,000,000 positive divisors.

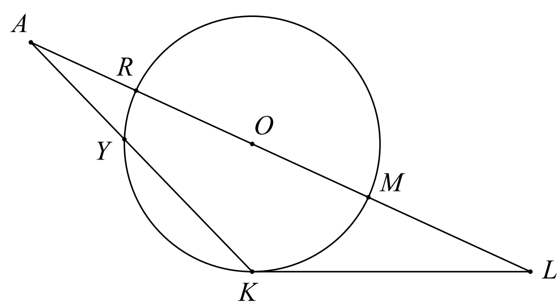
T-6. Given an arbitrary finite sequence of letters (represented as a word), a *subsequence* is a sequence of one or more letters that appear in the same order as in the original sequence. For example, N, CT, OTT, and $CONTEST$ are subsequences of the word $CONTEST$, but NOT, $ONSET$, and $TESS$ are not. Assuming the standard English alphabet $\{A, B, \ldots, Z\}$, compute the number of distinct four-letter "words" for which EE is a subsequence.

T-7. Six solid regular tetrahedra are placed on a flat surface so that their bases form a regular hexagon $\mathcal{H}$ with side length 1, and so that the vertices not lying in the plane of $\mathcal{H}$ (the "top" vertices) are themselves coplanar. A spherical ball of radius r is placed so that its center is directly above the center of the hexagon. The sphere rests on the tetrahedra so that it is tangent to one edge from each tetrahedron. If the ball's center is coplanar with the top vertices of the tetrahedra, compute r.
(The bases of the tetrahedra fill the interior of the hexagon.)

T-8. Derek starts at the point $(0, 0)$, facing the point $(0, 1)$, and he wants to get to the point $(1, 1)$. He takes unit steps parallel to the coordinate axes. A move consists of either a step forward, or a $90°$ right (clockwise) turn followed by a step forward, so that his path does not contain any left turns. His path is restricted to the square region defined by $0 \le x \le 17$ and $0 \le y \le 17$. Compute the number of ways he can get to $(1, 1)$ without returning to any previously visited point.

T-9. The equations $x^3 + Ax + 10 = 0$ and $x^3 + Bx^2 + 50 = 0$ have two roots in common. Compute the product of these common roots.

T-10. Points A and L lie outside circle ω, whose center is O, and $\overline{AL}$ contains diameter $\overline{RM}$, as shown below. Circle ω is tangent to $\overline{LK}$ at K. Also, $\overline{AK}$ intersects ω at Y, which is between A and K. If $KL = 3$, $ML = 2$, and $m\angle AKL - m\angle YMK = 90°$, compute $[AKM]$ (i.e., the area of $\triangle AKM$).



Team Round Solutions (ARML 2010)

T-1. Rearrange the terms in the first equation to yield $x^2 + 4x + 12 = 6y - y^2 + 24$, so that the two equations together yield $4y = 6y - y^2 + 24$, or $y^2 - 2y - 24 = 0$, from which $y = 6$ or $y = -4$. If $y = 6$, then $x^2 + 4x + 12 = 24$, from which $x = -6$ or $x = 2$. If $y = -4$, then $x^2 + 4x + 12 = -16$, which has no real solutions because $x^2 + 4x + 12 = (x + 2)^2 + 8 \geq 8$ for all real x. So there are two ordered pairs satisfying the system, namely $\mathbf{(-6, 6)}$ and $\mathbf{(2, 6)}$.

T-2. If $\log^*(a) = 3$, then $\log(\log(\log(a))) \leq 1$ and $\log(\log(a)) > 1$. If $\log(\log(a)) > 1$, then $\log(a) > 10$ and $a > 10^{10}$. Because the problem asks for the smallest such a that is an integer, choose $a = 10^{10} + 1 = 10{,}000{,}000{,}001$, which has **9** zeros.

T-3. If a number N has k base 10 digits, then its maximum point value is $(k-1)+(k-2)+\cdots+1 = \frac{1}{2}(k-1)(k)$. So if $k \leq 5$, the number N is worth at most 10 points. Therefore the desired number has at least six digits. If $100{,}000 < N < 101{,}000$, then N is of the form $100\underline{A}\,\underline{B}\,\underline{C}$, which could yield 12 possible primes, namely $1\underline{A}$, $1\underline{B}$, $1\underline{C}$, $0\underline{A}$ (twice), $0\underline{B}$ (twice), $0\underline{C}$ (twice), $\underline{A}\,\underline{B}$, $\underline{A}\,\underline{C}$, $\underline{B}\,\underline{C}$. So search for N of the form $100\underline{A}\,\underline{B}\,\underline{C}$, starting with lowest values first. Notice that if any of A, B, or C is not a prime, at least two points are lost, and so all three numbers must be prime. Proceed by cases:

First consider the case $A = 2$. Then $1\underline{A}$ is composite, so all of $1\underline{B}$, $1\underline{C}$, $\underline{A}\,\underline{B}$, $\underline{A}\,\underline{C}$, $\underline{B}\,\underline{C}$ must be prime. Considering now the values of $1\underline{B}$ and $1\underline{C}$, both B and C must be in the set $\{3, 7\}$. Because 27 is composite, $B = C = 3$, but then $\underline{B}\,\underline{C} = 33$ is composite. So A cannot equal 2.

If $A = 3$, then $B \neq 2$ because both 12 and 32 are composite. If $B = 3$, $1\underline{B}$ is prime but $\underline{A}\,\underline{B} = 33$ is composite, so all of C, $1\underline{C}$, and $3\underline{C}$ must be prime. These conditions are satisfied by $C = 7$ and no other value. So $A = B = 3$ and $C = 7$, yielding $N = \mathbf{100337}$.

T-4. Only two triangles have no sides that are sides of the original hexagon: $A_1A_3A_5$ and $A_2A_4A_6$. For each of these triangles, there are $2^3 - 1 = 7$ colorings in which at least one side is red, for a total of $7 \cdot 7 = 49$ colorings of those six diagonals. The colorings of the three central diagonals $\overline{A_1A_4}, \overline{A_2A_5}, \overline{A_3A_6}$ are irrelevant because the only triangles they can form include sides of the original hexagon, so they can be colored in $2^3 = 8$ ways, for a total of $8 \cdot 49 = \mathbf{392}$ colorings.

T-5. Let k denote the number of distinct prime divisors of n, so that $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, $a_i > 0$. Then if $d(x)$ denotes the number of positive divisors of x,

$$d(n^n) = (a_1 n + 1)(a_2 n + 1) \cdots (a_k n + 1) \geq (n + 1)^k. \qquad (*)$$

Note that if $n \geq 99$ and $k \geq 3$, then $d(n^n) \geq 100^3 = 10^6$, so $102 = 2 \cdot 3 \cdot 17$ is an upper bound for the solution. Look for values less than 99, using two observations: (1) all $a_i \leq 6$

(because $p^7 > 99$ for all primes); and (2) $k \leq 3$ (because $2 \cdot 3 \cdot 5 \cdot 7 > 99$). These two facts rule out the cases $k = 1$ (because $(*)$ yields $d \leq (6n+1)^1 < 601$) and $k = 2$ (because $d(n^n) \leq (6n+1)^2 < 601^2$).

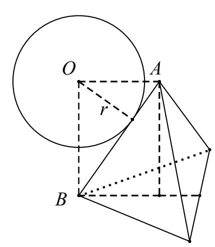
So $k = 3$. Note that if $a_1 = a_2 = a_3 = 1$, then from $(*)$, $d(n^n) = (n+1)^3 < 10^6$. So consider only $n < 99$ with exactly three prime divisors, and for which not all exponents are 1. The only candidates are 60, 84, and 90; of these, $n = 84$ is the smallest one that works:

$$\begin{aligned} d(60^{60}) &= d(2^{120} \cdot 3^{60} \cdot 5^{60}) = 121 \cdot 61 \cdot 61 < 125 \cdot 80 \cdot 80 = 800{,}000; \\ d(84^{84}) &= d(2^{168} \cdot 3^{84} \cdot 7^{84}) = 169 \cdot 85 \cdot 85 > 160 \cdot 80 \cdot 80 = 1{,}024{,}000. \end{aligned}$$

Therefore $n = \mathbf{84}$ is the least positive integer n such that $d(n^n) > 1{,}000{,}000$.

T-6. Divide into cases according to the number of E's in the word. If there are only two E's, then the word must have two non-E letters, represented by ?'s. There are $\binom{4}{2} = 6$ arrangements of two E's and two ?'s, and each of the ?'s can be any of 25 letters, so there are $6 \cdot 25^2 = 3750$ possible words. If there are three E's, then the word has exactly one non-E letter, and so there are 4 arrangements times 25 choices for the letter, or 100 possible words. There is one word with four E's, hence a total of **3851** words.

T-7. Let O be the center of the sphere, A be the top vertex of one tetrahedron, and B be the center of the hexagon.



Then BO equals the height of the tetrahedron, which is $\frac{\sqrt{6}}{3}$. Because A is directly above the centroid of the bottom face, AO is two-thirds the length of the median of one triangular face, so $AO = \frac{2}{3}\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{3}$. The radius of the sphere is the altitude to hypotenuse $\overline{AB}$ of $\triangle ABO$, so the area of $\triangle ABO$ can be represented in two ways: $[ABO] = \frac{1}{2}AO \cdot BO = \frac{1}{2}AB \cdot r$. Substitute given and computed values to obtain $\frac{1}{2}\left(\frac{\sqrt{3}}{3}\right)\left(\frac{\sqrt{6}}{3}\right) = \frac{1}{2}(1)(r)$, from which $r = \frac{\sqrt{18}}{9} = \boldsymbol{\frac{\sqrt{2}}{3}}$.

T-8. Divide into cases according to the number of right turns Derek makes.

- There is one route involving only one turn: move first to $(0, 1)$ and then to $(1, 1)$.
- If he makes two turns, he could move up to $(0, a)$ then to $(1, a)$ and then down to $(1, 1)$. In order to do this, a must satisfy $1 < a \leq 17$, leading to 16 options.

- If Derek makes three turns, his path is entirely determined by the point at which he turns for the second time. If the coordinates of this second turn point are (a, b), then both a and b are between 2 and 17 inclusive, yielding $(17-1)^2$ possibilities.
- If Derek makes four turns, his last turn must be from facing in the $-x$-direction to the $+y$-direction. For this to be his last turn, it must occur at $(1, 0)$. Then his next-to-last turn could be at any $(a, 0)$, with $1 < a \leq 17$, depending on the location of his second turn as in the previous case. This adds another $(17-1)^2$ possibilities.
- It is impossible for Derek to make more than four turns and get to $(1, 1)$ without crossing or overlapping his path.

Summing up the possibilities gives $1 + 16 + 16^2 + 16^2 = \mathbf{529}$ possibilities.

T-9. Let the roots of the first equation be p, q, r and the roots of the second equation be p, q, s. Then $pqr = -10$ and $pqs = -50$, so $\frac{s}{r} = 5$. Also $p + q + r = 0$ and $p + q + s = -B$, so $r - s = B$. Substituting yields $r - 5r = -4r = B$, so $r = -\frac{B}{4}$ and $s = -\frac{5B}{4}$. From the second given equation, $pq + ps + qs = pq + s(p + q) = 0$, so $pq - \frac{5B}{4}(p + q) = 0$, or $pq = \frac{5B}{4}(p + q)$. Because $p + q + r = 0$, $p + q = -r = \frac{B}{4}$, and so $pq = \frac{5B^2}{16}$. Because $pqr = -10$ and $r = -\frac{B}{4}$, conclude that $pq = \frac{40}{B}$. Thus $\frac{5B^2}{16} = \frac{40}{B}$, so $B^3 = 128$ and $B = 4\sqrt[3]{2}$. Then $pq = \frac{5B^2}{16}$ implies that $pq = \mathbf{5\sqrt[3]{4}}$ (and $r = -\sqrt[3]{2}$).

Alternate Solution: Let the common roots be p and q. Then the following polynomials (linear combinations of the originals) must also have p and q as common zeros:

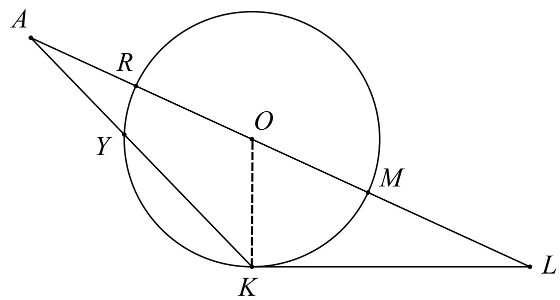
$$\begin{aligned}(x^3 + Bx^2 + 50) - (x^3 + Ax + 10) &= Bx^2 - Ax + 40 \\ -(x^3 + Bx^2 + 50) + 5(x^3 + Ax + 10) &= 4x^3 - Bx^2 + 5Ax.\end{aligned}$$

Because $pq \neq 0$, neither p nor q is zero, so the second polynomial has zeros p, q, and 0. Therefore p and q are zeros of $4x^2 - Bx + 5A$. [This result can also be obtained by using the Euclidean Algorithm on the original polynomials.]

Because the two quadratic equations have the same zeros, their coefficients are proportional: $\frac{4}{B} = \frac{5A}{40} \Rightarrow AB = 32$ and $\frac{4}{B} = \frac{-B}{-A} \Rightarrow 4A = B^2$. Hence $\frac{128}{B} = B^2$ and $B^3 = 128$, so $B = 4\sqrt[3]{2}$. Rewriting the first quadratic as $B\left(x^2 - \frac{A}{B}x + \frac{40}{B}\right)$ shows that the product $pq = \frac{40}{B} = \mathbf{5\sqrt[3]{4}}$.

Alternate Solution: Using the sum of roots formulas, notice that $pq+ps+qs = p+q+r = 0$. Therefore $0 = pq + ps + qs - (p + q + r)s = pq - rs$, and $pq = rs$. Hence $(pq)^3 = (pqr)(pqs) = (-10)(-50) = 500$, so $pq = \mathbf{5\sqrt[3]{4}}$.

T-10. Notice that $\overline{OK} \perp \overline{KL}$, and let r be the radius of ω.



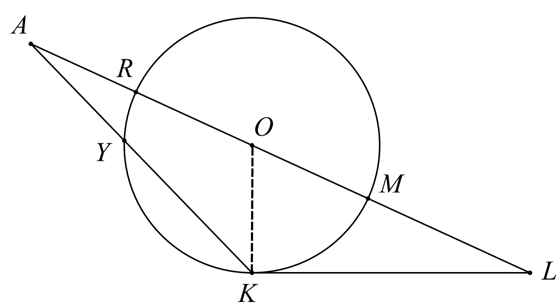
Then consider right triangle OKL. Because $ML = 2$, $OK = r$, and $OL = r + 2$, it follows that $r^2 + 3^2 = (r+2)^2$, from which $r = \frac{5}{4}$.

Because $\mathrm{m}\angle YKL = \frac{1}{2}\mathrm{m}\widehat{YRK}$ and $\mathrm{m}\angle YMK = \frac{1}{2}\mathrm{m}\widehat{YK}$, it follows that $\mathrm{m}\angle YKL + \mathrm{m}\angle YMK = 180°$. By the given condition, $\mathrm{m}\angle YKL - \mathrm{m}\angle YMK = 90°$. It follows that $\mathrm{m}\angle YMK = 45°$ and $\mathrm{m}\angle YKL = 135°$. hence $\mathrm{m}\widehat{YK} = 90°$. Thus,

$$\overline{YO} \perp \overline{OK} \quad \text{and} \quad \overline{YO} \parallel \overline{KL}. \tag{$*$}$$

From here there are several solutions:

First Solution: Compute $[AKM]$ as $\frac{1}{2}\text{base} \cdot \text{height}$, using base $\overline{AM}$.



Because of $(*)$, $\triangle AYO \sim \triangle AKL$. To compute AM, notice that in $\triangle AYO$, $AO = AM - r$, while in $\triangle AKL$, the corresponding side $AL = AM + ML = AM + 2$. Therefore:

$$\begin{aligned} \frac{AO}{AL} &= \frac{YO}{KL} \\ \frac{AM - \frac{5}{4}}{AM + 2} &= \frac{5/4}{3}, \end{aligned}$$

from which $AM = \frac{25}{7}$. Draw the altitude of $\triangle AKM$ from vertex K, and let h be its length. In right triangle OKL, h is the altitude to the hypotenuse, so $\frac{h}{3} = \sin(\angle KLO) = \frac{r}{r+2}$. Hence $h = \frac{15}{13}$. Therefore $[AKM] = \frac{1}{2} \cdot \frac{25}{7} \cdot \frac{15}{13} = \mathbf{\frac{375}{182}}$.

Second Solution: By the Power of the Point Theorem, $LK^2 = LM \cdot LR$, so

$$\begin{aligned} LR &= \frac{9}{2}, \\ RM = LR - LM &= \frac{5}{2}, \\ OL = r + ML &= \frac{13}{4}. \quad (\dagger) \end{aligned}$$

From $(*)$, we know that $\triangle AYO \sim \triangle AKL$. Hence by $(\dagger)$,

$$\frac{AL}{AO} = \frac{AL}{AL - OL} = \frac{KL}{YO} = \frac{3}{5/4} = \frac{12}{5}, \quad \text{thus} \quad AL = \frac{12}{7} \cdot OL = \frac{12}{7} \cdot \frac{13}{4} = \frac{39}{7}.$$

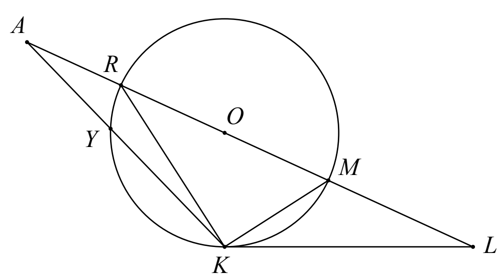
Hence $AM = AL - 2 = \frac{25}{7}$. The ratio between the areas of triangles AKM and RKM is equal to

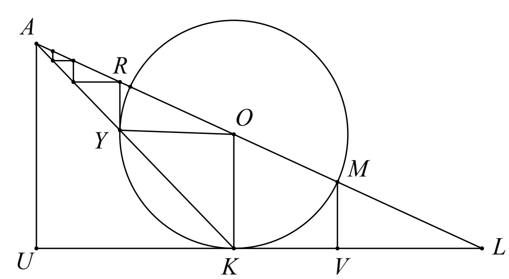
$$\frac{[AKM]}{[RKM]} = \frac{AM}{RM} = \frac{25/7}{5/2} = \frac{10}{7}.$$

Thus $[AKM] = \frac{10}{7} \cdot [RKM]$.

Because $\angle KRL$ and $\angle MKL$ both subtend $\widehat{KM}$, $\triangle KRL \sim \triangle MKL$. Therefore $\frac{KR}{MK} = \frac{LK}{LM} = \frac{3}{2}$. Thus let $KR = 3x$ and $MK = 2x$ for some positive real number x. Because RM is a diameter of ω (see left diagram below), m$\angle RKM = 90°$. Thus triangle RKM is a right triangle with hypotenuse $\overline{RM}$. In particular, $13x^2 = KR^2 + MK^2 = RM^2 = \frac{25}{4}$, so $x^2 = \frac{25}{52}$ and $[RKM] = \frac{RK \cdot KM}{2} = 3x^2$. Therefore

$$[AKM] = \frac{10}{7} \cdot [RKM] = \frac{10}{7} \cdot 3 \cdot \frac{25}{52} = \mathbf{\frac{375}{182}}.$$





Third Solution: Let U and V be the respective feet of the perpendiculars dropped from A and M to $\overleftrightarrow{KL}$. From $(*)$, $\triangle AKL$ can be dissected into two infinite progressions of triangles: one progression of triangles similar to $\triangle OKL$ and the other similar to $\triangle YOK$, as shown in the right diagram above. In both progressions, the corresponding sides of the triangles have common ratio equal to

$$\frac{YO}{KL} = \frac{5/4}{3} = \frac{5}{12}.$$

Thus

$$AU = \frac{5}{4}\left(1 + \frac{5}{12} + \left(\frac{5}{12}\right)^2 + \cdots\right) = \frac{5}{4} \cdot \frac{12}{7} = \frac{15}{7}.$$

Because $\triangle LMV \sim \triangle LOK$, and because $LO = \frac{13}{4}$ by (†),

$$\frac{MV}{OK} = \frac{LM}{LO}, \quad \text{thus} \quad MV = \frac{OK \cdot LM}{LO} = \frac{\frac{5}{4} \cdot 2}{\frac{13}{4}} = \frac{10}{13}.$$

Finally, note that $[AKM] = [AKL] - [KLM]$. Because $\triangle AKL$ and $\triangle KLM$ share base $\overline{KL}$,

$$[AKM] = \frac{1}{2} \cdot 3 \cdot \left(\frac{15}{7} - \frac{10}{13}\right) = \mathbf{\frac{375}{182}}.$$