Note: Calculators and the Internet were allowed in this meet.

April 7, 2022
Round 1: Arithmetic and Number Theory

1. For integers m and n, let the operations $*$ and $\oplus$ be defined by $m * n = m^2 + n^2$ and $m \oplus n = 3m + 2n$. Find $(\{5,6,7,8\} * 3) * (8 \oplus 7)$.

2. Find the number of perfect cubes between 1 and {800,000; 700,000; 500,000; 400,000} inclusive that are multiples of 12.

3. Fifteen Yankees fans, {14, 10, 12, 11} Red Sox fans, and Joanne will stand in an orderly line at the stadium. If these {30, 26, 28, 27} people are arranged in the line at random, then the probability that Joanne is directly behind a Red Sox fan and directly in front of a Yankees fan is $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

April 7, 2022
Round II: Algebra I (Real numbers and no transcendental functions)

1. Suppose that $\frac{b}{a} = \{3,4,5,6\}$ and $\frac{c}{b} = \{5,6,7,8\}$. Then $\frac{a+b}{b+c} = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

2. Line l_1 has slope 3 and intersects the x-axis at A. Line l_2 has slope $\{-1.1, -1.2, -1.3, -1.4\}$ and intersects the x-axis at B. The point of intersection of l_1 and l_2 lies on the line $y = 157$. Find the distance AB, giving your answer to the nearest integer.

3. Let f be the function defined by $f(x) = \left|x + \frac{1}{2}\right| + \left|x - \frac{1}{2}\right|$ and let $f^2(x) = f\big(f(x)\big)$, $f^3(x) = f\big(f^2(x)\big)$, and so on. Find the area enclosed by the graph of $y = f^8(x)$ and the line $y = \{768, 1792, 2304, 1280\}$.

April 7, 2022
Round III: Geometry (figures are not drawn to scale)

1. The volume of a rectangular solid *A* is 250. Each dimension of *A* is increased by {80%, 60%, 40%, 20%} to form a new solid *B*. Find the volume of *B*.

2. The points *A*(0, 0), *B*({8,6,4,10}, 0), *C*({7,5,3,9}, 1), *D*(1, 1) are given. Suppose that the line $y = 2(x - k)$ divides quadrilateral *ABCD* into two equal areas. Then $k = \frac{a}{b}$, where *a* and *b* are relatively prime positive integers. Find $a + b$.

3. Let *ABCD* be a square, with $AB = 1$. Let *M* be the midpoint of side $\overline{AB}$ and let *N* be that point on side $\overline{DC}$ such that $DN = \left\{\frac{1}{9}, \frac{1}{5}, \frac{2}{7}, \frac{1}{7}\right\} DC$. Let *P* be the point of intersection of $\overline{MN}$ and $\overline{AC}$. The area of quadrilateral *MBCP* is $\frac{a}{b}$, where *a* and *b* are relatively prime positive integers. Find $a + b$.

April 7, 2022
Round IV: Algebra II

1. For how many integers n, with $1 \leq n \leq 10{,}000$, is $\log_{\{5,6,7,8\}} n$ an integer?

2. Suppose that the polynomial $x^2 - \{3x, 2x, 4x, x\} + 5$ divides the polynomial $x^3 + ax^2 + bx - 15$, where a and b are constants. Find $a^2 + b^2$.

3. Find the integer k, with $\{5000 \leq k \leq 6000,\ 9000 \leq k \leq 10000,\ 1000 \leq k \leq 2000,\ 2000 \leq k \leq 3000\}$, for which $x^4 + k$ can be factored into two distinct trinomial factors with integer coefficients.

April 7, 2022
Round V: Analytic Geometry

1. If the points $(1, p)$ and $(-1, q)$ lie on the graph of $y = ax^2 + bx + c$ and $p - q = \{6,8,10,12\}$, what is the value of b ?

2. The points $A(0, 0)$, $B(9, 3)$, and $P(0, b)$, $b > 0$, are given. Let Q and R be the points on line segment AB such that $AQ = RB = \frac{1}{3}AB$. Suppose that (slope of $\overline{PR}$) $= \{\frac{2}{7}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}\}$(slope of $\overline{PQ}$). Then $b = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

3. For any two real numbers p and q, the points $P(\{5,3,5,3\}p^2, \{10,6,10,6\}p)$ and $Q(\{5,3,5,3\}q^2, \{10,6,10,6\}q)$ lie on the parabola C_1 with equation $y^2 = \{20,12,20,12\}x$. Suppose now that p and q vary in such a way that pq always equals -1. Then the midpoint of line segment PQ traces out a different parabola, C_2. The distance between the vertex and the focus of C_2 is $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

April 7, 2022
Round VI: Trigonometry and Complex Numbers

1. A vertical pole stands on horizontal ground. An ant is at ground level, 6 meters from the base of the pole. If the ant looks at a point *A* on the pole, the angle of elevation is 50°. If the ant looks at a point *B* on the pole, the angle of elevation is {65°, 67°, 69°, 63°}. Find the distance *AB* in centimeters, to the nearest centimeter. (Do not attempt to include a unit in your answer.)

2. For two acute angles θ and ϕ, $\sin\theta = \cos\phi$. If $\theta = 4k + 22$ degrees and $\phi = 6k + \{13,12,14,10\}$ degrees, then $k = \frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

3. Suppose that *z* is a complex number such that $|z| = 1$ and $\arg z = \theta$ degrees, where $0 \leq \theta < 180$. For how many integer values of θ is $(1+z)^{\{75,48,80,135\}}$ a real number?

April 7, 2022
Team Round

1. Find the sum of all integers between 100 and 999 whose digits are all even.

2. Water pipe A fills a storage tank in 8 hours. Water pipe B fills the same storage tank in 12 hours. The tank is empty at time $t = 0$ hours, and at that time pipe A is turned on. Pipe B is turned on at $t = 1.5$ hours (with pipe A continuing to pour water into the tank). The tank becomes full at time $t = \frac{a}{b}$ hours, where a and b are relatively prime positive integers. Find $a + b$.

3. A frustum of a cone has bases of radii 6 and 8 and a slant height of 5. The volume of the frustum is $\frac{\pi a\sqrt{b}}{c}$, where a, b, c are positive integers, a and c are relatively prime, and b is not divisible by the square of any prime number. Find $a + b + c$.
(Terminology: A frustum of a cone is the part of the cone lying below a plane parallel to the base of the cone. The slant height of the frustum is the smallest possible length of a line segment from a point on the perimeter of one base to a point on the perimeter of the other base.)

4. For how many real numbers x with $1 \leq x \leq 1{,}000{,}000$ is $\log_2 x + \log_3 x$ an integer?

5. Suppose that points A and B lie in two-dimensional space and that $AB = 16$. Let the curve consisting of all points P such that $PA + PB = 20$ be E. Let R be the point on line segment AB such that $RB = 3$, and let l be the line through R perpendicular to $\overline{AB}$. Line l intersects E at points S and T. Find $100(ST)$ to the nearest integer.

6. How many times does the graph of $y = \sin\left(\frac{1}{x}\right)$ intersect the x-axis between $x = 0.002$ and $x = 1$?
(Note: Since no unit of angles is given, the quantity $\frac{1}{x}$ is interpreted as being in radians.)

Answers

Round 1

1. 2600, 3469, 4808, 6773
2. 15, 14, 13, 12
3. 36, 16, 26, 289

Round 2

1. 11, 33, 23, 61
2. 195, 183, 173, 164
3. 2240, 12480, 20672, 6336

Round 3

1. 1458, 1024, 686, 432
2. 19, 15, 11, 23
3. 141, 73, 95, 107

Round 4

1. 6, 6, 5, 5
2. 232, 146, 338, 80
3. 5184, 9604, 1024, 2500

Round 5

1. 3, 4, 5, 6
2. 13, 11, 17, 23
3. 7, 5, 7, 5

Round 6

1. 572, 698, 848, 463
2. 13, 33, 32, 34
3. 8, 12, 20, 23

Team Round

1. 54400
2. 32
3. 172
4. 33
5. 1039
6. 159