CSAML 2022

Note: In each case, the solution given is for the first version of the question.
Calculators and Internet were allowed in this meet.

April 7, 2022
Round 1: Arithmetic and Number Theory

1. For integers m and n, let the operations $*$ and $\oplus$ be defined by
$m * n = m^2 + n^2$ and $m \oplus n = 3m + 2n$. Find $(\{5,6,7,8\} * 3) * (8 \oplus 7)$.
[Answer: {2600, 3469, 4808, 6773}]

<u>Solution:</u>
$(5 * 3) * (8 \oplus 7) = 34 * 38 = 2600$.

2. Find the number of perfect cubes between 1 and {800,000; 700,000; 500,000; 400,000} inclusive that are multiples of 12.
[Answer: {15, 14, 13, 12}]

<u>Solution:</u>
Since $12 = 2^2 \cdot 3$, n^3 is a multiple of 12 if and only if n is a multiple of both 2 and 3; that is, n is a multiple of 6. For $1 \leq n^3 \leq 800000$, we need $1 \leq n \leq 92.83$, and the number of multiples of 6 between 1 and 92 is 15.

3. Fifteen Yankees fans, {14, 10, 12, 11} Red Sox fans, and Joanne will stand in an orderly line at the stadium. If these {30, 26, 28, 27} people are arranged in the line at random, then the probability that Joanne is directly behind a Red Sox fan and directly in front of a Yankees fan is $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.
[Answer: {36, 16, 26, 289}]

<u>Solution:</u>
Firstly, Joanne must be in place 2, 3, … , or 29 in the line. The probability that this happens is 28/30. If this is the case, the probability that the person directly in front of Joanne is a Red Sox fan is 14/29. If this is the case, the probability that the person directly behind Joanne is a Yankees fan is 15/28. So the required probability is $\frac{28}{30} \cdot \frac{14}{29} \cdot \frac{15}{28} = \frac{7}{29}$. So, the answer is 7 + 29 = 36.

April 7, 2022
Round II: Algebra I (Real numbers and no transcendental functions)

1. Suppose that $\frac{b}{a} = \{3,4,5,6\}$ and $\frac{c}{b} = \{5,6,7,8\}$. Then $\frac{a+b}{b+c} = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
[Answer: {11, 33, 23, 61}]

Solution:
$b = 3a$ and $c = 5b = 15a$. So, $\frac{a+b}{b+c} = \frac{a+3a}{3a+15a} = \frac{4a}{18a} = \frac{2}{9}$. So, the answer is 2 + 9 = 11.

2. Line l_1 has slope 3 and intersects the *x*-axis at *A*. Line l_2 has slope {−1.1,−1.2,−1.3,−1.4} and intersects the *x*-axis at *B*. The point of intersection of l_1 and l_2 lies on the line $y = 157$. Find the distance *AB*, giving your answer to the nearest integer.
[Answer: {195, 183, 173, 164}]

Solution:
Let the point of intersection of l_1 and l_2 be *P* and suppose that the line through *P* perpendicular to the *x*-axis intersects the *x*-axis at *Q*. Then $\frac{157}{AQ} = 3$ and $\frac{157}{QB} = 1.1$. So, $AB = AQ + QB$ $= \frac{157}{3} + \frac{157}{1.1} = 195.06$. So, the answer is 195.

3. Let f be the function defined by $f(x) = \left|x + \frac{1}{2}\right| + \left|x - \frac{1}{2}\right|$ and let $f^2(x) = f\big(f(x)\big)$, $f^3(x) = f\big(f^2(x)\big)$, and so on. Find the area enclosed by the graph of $y = f^8(x)$ and the line $y = \{768, 1792, 2304, 1280\}$.
[Answer: {2240, 12480, 20672, 6336}]

Solution:
First, for $x \geq \frac{1}{2}$, $f(x) = 2x$; for $-\frac{1}{2} < x < \frac{1}{2}$, $f(x) = 1$; and for $x \leq -\frac{1}{2}$, $f(x) = -2x$.
Also note that f is an even function. Now, for $x \geq \frac{1}{2}$, $f(x)$ is also $\geq \frac{1}{2}$, so $f^8(x) = 2^8 x = 256x$.
For $0 \leq x \leq \frac{1}{2}$, $f(x) = 1$, which is $\geq \frac{1}{2}$, so $f^2(x) = 2$, $f^3(x) = 2^2, \ldots, f^8(x) = 2^7 = 128$.
Also, since f is an even function, f^8 is an even function. Thus, for $x \geq \frac{1}{2}$, $f^8(x) = 256x$; for $-\frac{1}{2} < x < \frac{1}{2}$, $f^8(x) = 128$; and for $x \leq -\frac{1}{2}$, $f(x) = -256x$. The line $y = 768$ intersects the graph of $y = f^8(x)$ at (3, 768) and (−3, 768). So, the required region is a trapezoid whose area is $\frac{1}{2}(1 + 6) \cdot (768 - 128) = 2240$.

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April 7, 2022
Round III: Geometry (figures are not drawn to scale)

1. The volume of a rectangular solid *A* is 250. Each dimension of *A* is increased by {80%, 60%, 40%, 20%} to form a new solid *B*. Find the volume of *B*.
[Answer: {1458, 1024, 686, 432}]

<u>Solution:</u>
Volume of *B* = $250(1.8)^3 = 1458$

2. The points *A*(0, 0), *B*({8,6,4,10}, 0), *C*({7,5,3,9}, 1), *D*(1, 1) are given. Suppose that the line $y = 2(x - k)$ divides quadrilateral *ABCD* into two equal areas. Then $k = \frac{a}{b}$, where *a* and *b* are relatively prime positive integers. Find $a + b$.
[Answer: {19, 15, 11, 23}]

<u>Solution:</u>
The given quadrilateral is an isosceles trapezoid, and so, by symmetry, in order to divide the quadrilateral into two equal areas the given line must pass through the midpoint of the median of the trapezoid, which is the point (4, ½). So, $\frac{1}{2} = 2(4 - k)$, which gives $k = \frac{15}{4}$. So, the answer is 15 + 4 = 19.

3. Let *ABCD* be a square, with $AB = 1$. Let *M* be the midpoint of side $\overline{AB}$ and let *N* be that point on side $\overline{DC}$ such that $DN = \left\{\frac{1}{9}, \frac{1}{5}, \frac{2}{7}, \frac{1}{7}\right\} DC$. Let *P* be the point of intersection of $\overline{MN}$ and $\overline{AC}$. The area of quadrilateral *MBCP* is $\frac{a}{b}$, where *a* and *b* are relatively prime positive integers. Find $a + b$.
[Answer: {141, 73, 95, 107}]

<u>Solution:</u>
Draw square *ABCD* with *A* in the bottom left corner and *B* to the right of *A*. Set up a system of axes so that the origin is at *A*, line *AB* is coincident with the *x*-axis, and line *AD* is coincident with the *y*-axis. Then we have points $M\left(\frac{1}{2}, 0\right)$ and $N\left(\frac{1}{9}, 1\right)$. It follows that the equations of lines *AC* and *MN* are, respectively, $y = x$ and $y = -\frac{18}{7}\left(x - \frac{1}{2}\right)$. The *y*-coordinate of the point of intersection *P* of these two lines is then $\frac{9}{25}$. So, using brackets for areas, $[AMP] = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{9}{25} = \frac{9}{100}$. Thus, $[MBCP] = [ABC] - [AMP] = \frac{1}{2} - \frac{9}{100} = \frac{41}{100}$. So, the answer is 141.

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Round IV: Algebra II

1. For how many integers n, with $1 \le n \le 10{,}000$, is $\log_{\{5,6,7,8\}} n$ an integer?
[Answer: {6, 6, 5, 5}]

<u>Solution:</u>
First note that $\log_5 10000 = 5.72$. So, the required values of n are $5^0, 5^1, 5^2, \dots, 5^5$. Thus, there are 6 such values of n.

2. Suppose that the polynomial $x^2 - \{3x, 2x, 4x, x\} + 5$ divides the polynomial $x^3 + ax^2 + bx - 15$, where a and b are constants. Find $a^2 + b^2$.
[Answer: {232, 146, 338, 80}]

<u>Solution:</u>
Suppose that $(x^2 - 3x + 5)(px + q) = x^3 + ax^2 + bx - 15$. Looking at the leading coefficient on the right we see that $p = 1$, and looking at the constant term on the right we see that $5q = -15$, telling us that $q = -3$. But $(x^2 - 3x + 5)(x - 3) = x^3 - 6x^2 + 14x - 15$. So $a = -6$ and $b = 14$, making $a^2 + b^2 = 232$.

3. Find the integer k, with $\{5000 \le k \le 6000, 9000 \le k \le 10000, 1000 \le k \le 2000, 2000 \le k \le 3000\}$, for which $x^4 + k$ can be factored into two distinct trinomial factors with integer coefficients.
[Answer: {5184, 9604, 1024, 2500}]

<u>Solution:</u>
Suppose that $x^4 + k = (x^2 + a_1x + b_1)(x^2 + a_2x + b_2)$, where all the coefficients given are integers. The right-hand side is
$x^4 + (a_1 + a_2)x^3 + (b_1 + a_1a_2 + b_2)x^2 + (a_1b_2 + a_2b_1)x + b_1b_2$. So, equating coefficients, we obtain three equations:
(1) $a_1 + a_2 = 0$
(2) $b_1 + a_1a_2 + b_2 = 0$
(3) $a_1b_2 + a_2b_1 = 0$
(4) $b_1b_2 = k$
By equation (1), $a_1 = -a_2 = a$, say. This equation tells us that a_1 and a_2 have opposite signs, and so we can let a_1 be the nonnegative one of the two. So $a \ge 0$.
Suppose $a = 0$. Then, by (2), $b_1 = -b_2$, and then, by (4), $k < 0$, which is not the case. So $a > 0$. Then, by (3), $b_1 = b_2 = b$, say.
Now, by (2), $a^2 = 2b$, and so a is even. So we can say $a = 2m$, where m is a positive integer. Then $2b = a^2 = 4m^2$, so $b = 2m^2$. Finally, by (4), $k = b^2 = 4m^4$. So $k = 4m^4$.
Now we know that $5000 \le k \le 6000$. So $1250 \le 4m^4 \le 1500$. So $1250 \le m^4 \le 1500$. So $5.95 \le m \le 6.22$, telling us that $m = 6$. So $k = 4 \cdot 6^4 = 5184$.

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Round V: Analytic Geometry

1. If the points $(1, p)$ and $(-1, q)$ lie on the graph of $y = ax^2 + bx + c$ and $p - q = \{6,8,10,12\}$, what is the value of b ?
[Answer: {3, 4, 5, 6}]

<u>Solution:</u>
Substituting the given coordinates into the equation we get $p = a + b + c$ and $q = a - b + c$. Subtracting, $p - q = 2b$. So $6 = 2b$, telling us that $b = 3$.

2. The points $A(0, 0)$, $B(9, 3)$, and $P(0, b)$, $b > 0$, are given. Let Q and R be the points on line segment AB such that $AQ = RB = \frac{1}{3}AB$. Suppose that $(\text{slope of } \overline{PR}) = \{\frac{2}{7}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}\}(\text{slope of } \overline{PQ})$. Then $b = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
[Answer: {13, 11, 17, 23}]

<u>Solution:</u>
The points Q and R are (3, 1) and (6, 2), respectively. So, the given slope equation translates to $\frac{b-2}{0-6} = \frac{2}{7}\left(\frac{b-1}{0-3}\right)$. Solving for b we get $b = \frac{10}{3}$. So the answer is 13.

3. For any two real numbers p and q, the points $P(\{5,3,5,3\}p^2, \{10,6,10,6\}p)$ and $Q(\{5,3,5,3\}q^2, \{10,6,10,6\}q)$ lie on the parabola C_1 with equation $y^2 = \{20,12,20,12\}x$. Suppose now that p and q vary in such a way that pq always equals -1. Then the midpoint of line segment PQ traces out a different parabola, C_2. The distance between the vertex and the focus of C_2 is $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.
[Answer: {7, 5, 7, 5}]

<u>Solution:</u>
The midpoint of PQ has coordinates $\left(\frac{5}{2}(p^2 + q^2), 5(p + q)\right)$. Letting the x-coordinate of this variable point be x and the y-coordinate be y, we get
$y^2 = 25(p^2 + q^2 + 2pq) = 25(p^2 + q^2 - 2) = 25(p^2 + q^2) - 50 = 25 \cdot \frac{2x}{5} - 50 = 10x - 50$
$= 10(x - 5)$. So the parabola traced out by midpoint has equation $y^2 = 10(x - 5)$. The distance between the vertex and the focus for this parabola is the same as the equivalent distance for the parabola $y^2 = 10x$. Recall that the distance between the vertex and the focus of the parabola $y^2 = 4ax$ is a. Here $a = \frac{5}{2}$. So the answer to the question is $5 + 2 = 7$.

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Round VI: Trigonometry and Complex Numbers

1. A vertical pole stands on horizontal ground. An ant is at ground level, 6 meters from the base of the pole. If the ant looks at a point A on the pole, the angle of elevation is 50°. If the ant looks at a point B on the pole, the angle of elevation is {65°, 67°, 69°, 63°}. Find the distance AB in centimeters, to the nearest centimeter. (Do not attempt to include a unit in your answer.)
[Answer: {572, 698, 848, 463}]

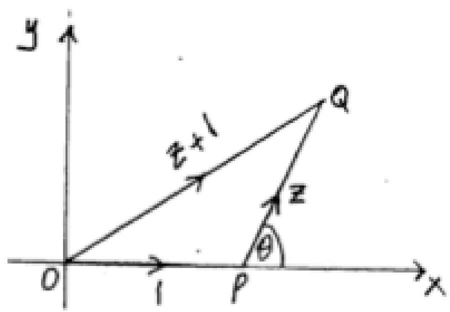
Solution:
Let the base of the pole be P. Then $\frac{AP}{6} = \tan 50°$ and $\frac{BP}{6} = \tan 65°$. So $AB = BP - AP$ $= 6(\tan 65° - \tan 50°) = 5.7165$ meters. So, to the nearest centimeter, AB = 572 centimeters.

2. For two acute angles θ and ϕ, $\sin\theta = \cos\phi$. If $\theta = 4k + 22$ degrees and $\phi = 6k + \{13,12,14,10\}$ degrees, then $k = \frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.
[Answer: {13, 33, 32, 34}]

Solution:
Since the angles are acute, $\theta + \phi = 90$ degrees. So $(4k + 22) + (6k + 13) = 90$. From this we get $k = \frac{11}{2}$. So the answer is 11 + 2 = 13.

3. Suppose that z is a complex number such that $|z| = 1$ and $\arg z = \theta$ degrees, where $0 \leq \theta < 180$. For how many integer values of θ is $(1 + z)^{\{75,48,80,135\}}$ a real number?
[Answer: {8, 12, 20, 23}]

Solution:



The diagram above shows the complex numbers 1, z, and $z + 1$. Since $|z| = 1$, we have $OP = PQ = 1$. Hence, the triangle OPQ is isosceles. From this it is easy to see that $\arg(1 + z) = \frac{\theta}{2}$. So $\arg(1 + z)^{75} = 75 \cdot \frac{\theta}{2}$. Thus, for $(1 + z)^{75}$ to be real we need $\frac{75\theta}{2} = 180k$, where k is an integer. (Note that, since $0 \leq \theta < 180$, we have $0 \leq k < 37.5$.) Hence $\theta = \frac{360k}{75} = \frac{24k}{5}$. So, for θ to be an integer we need k to be a multiple of 5. So $k = 0, 5, 10, \ldots, 35$. This is 8 possible values of k.

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April 7, 2022
Team Round

1. Find the sum of all integers between 100 and 999 whose digits are all even.
[Answer: 54400]

Solution:
Imagine that we list all of these numbers. We will sum the contributions of the digits firstly looking at the contributions of all 2's, then all 4's, and so on.

The number 2 will appear in the 100s place 5·5 times, counting for 200 each time.
The number 2 will appear in the 10s place 5·4 times, counting for 20 each time.
The number 2 will appear in the units place 5·4 times, counting for 2 each time.
So, the total contribution from 2s is 2(100·5·5 + 10·5·4 + 5·4) = 2(2720).

Continuing in the same way for 4, 6, and 8, we see that the required sum is
(2 + 4 + 6 + 8)(2720) = 54400.

2. Water pipe A fills a storage tank in 8 hours. Water pipe B fills the same storage tank in 12 hours. The tank is empty at time $t = 0$ hours, and at that time pipe A is turned on. Pipe B is turned on at $t = 1.5$ hours (with pipe A continuing to pour water into the tank). The tank becomes full at time $t = \frac{a}{b}$ hours, where a and b are relatively prime positive integers. Find $a + b$.
[Answer: 32]

Solution:
Pipe A fills at a rate of 1/8 tanks per hour, and B fills at a rate of 1/12 tanks per hour. Once both of the pipes are active, A has been working for t hours and B has been working for $t - 1.5$ hours. So, the proportion of the tank that is filled at time t is $\frac{1}{8}t + \frac{1}{12}(t - 1.5)$. Thus, at the moment the tank is full we have $\frac{1}{8}t + \frac{1}{12}(t - 1.5) = 1$. This gives $t = \frac{27}{5}$, so the answer is $27 + 5 = 32$.

3. A frustum of a cone has bases of radii 6 and 8 and a slant height of 5. The volume of the frustum is $\frac{\pi a\sqrt{b}}{c}$, where a, b, c are positive integers, a and c are relatively prime, and b is not divisible by the square of any prime number. Find $a + b + c$.
(Terminology: A frustum of a cone is the part of the cone lying below a plane parallel to the base of the cone. The slant height of the frustum is the smallest possible length of a line segment from a point on the perimeter of one base to a point on the perimeter of the other base.)
[Answer: 172]

Solution:
Let the height of the entire cone be H and let the height of the removed cone be h. Using the Pythagorean theorem, we find that the height of the frustum is $\sqrt{5^2 - (8-6)^2} = \sqrt{21}$. So $H - h = \sqrt{21}$. By similar triangles we have $\frac{h}{H} = \frac{6}{8} = \frac{3}{4}$. Solving this system of equations, we get $H = 4\sqrt{21}$ and $h = 3\sqrt{21}$. Thus, the volume of the frustum is $\frac{1}{3}\pi\left(8^2 \cdot 4\sqrt{21} - 6^2 \cdot 4\sqrt{21}\right)$ $= \frac{148\pi\sqrt{21}}{3}$. So, the answer to the question is $148 + 21 + 3 = 172$.

4. For how many real numbers x with $1 \le x \le 1{,}000{,}000$ is $\log_2 x + \log_3 x$ an integer?
[Answer: 33]

Solution:
Let $f(x) = \log_2 x + \log_3 x$. Then f is an increasing, continuous function. Note that $f(1) = 0$ and $f(1{,}000{,}000) = 32.51$. So the range of f is [0, 32.51], and f is a one-to-one function. There are 33 integers in this range, and since f is one-to-one, the number *x*-values so that $f(x)$ is an integer is 33.

5. Suppose that points *A* and *B* lie in two-dimensional space and that *AB* = 16. Let the curve consisting of all points *P* such that $PA + PB = 20$ be *E*. Let *R* be the point on line segment *AB* such that $RB = 3$, and let *l* be the line through *R* perpendicular to $\overline{AB}$. Line *l* intersects *E* at points *S* and *T*. Find $100(ST)$ to the nearest integer.
[Answer: 1039]

Solution:
Set up a coordinate system by placing the origin at the midpoint of line segment *AB* and making the *x*-axis be coincident with the line *AB*. Then we have points *A*(−8, 0) and *B*(8, 0). Using the sum-of-focal-distances definition of an ellipse we see that *E* is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with $a = \frac{20}{2} = 10$ and $c = 8$ (using the standard convention of *c* being the distance between the center and a focus of the ellipse). It is known for an ellipse that $a^2 = b^2 + c^2$, so $b^2 = a^2 - c^2 = 10^2 - 8^2 = 36$. So the equation of E is $\frac{x^2}{100} + \frac{y^2}{36} = 1$. Point R is (5, 0) and line *l* is $x = 5$, so the *y*-coordinates of S and T are given by the equation $\frac{5^2}{100} + \frac{y^2}{36} = 1$. This tells us that the *y*-coordinates of S and T are $\pm 3\sqrt{3}$. So $100(ST) = 100 \cdot 6\sqrt{3} = 1039.23$, and the answer to the question is 1039.

6. How many times does the graph of $y = \sin\left(\frac{1}{x}\right)$ intersect the *x*-axis between $x = 0.002$ and $x = 1$?
(Note: Since no unit of angles is given, the quantity $\frac{1}{x}$ is interpreted as being in radians.)
[Answer: 159]

Solution:
The graph intersects the *x*-axis when $\sin\left(\frac{1}{x}\right) = 0$, and this happens when $\frac{1}{x} = k\pi$ for some integer *k*. Now $0.002 \le x \le 1$, so $1 \le x \le 500$, so we need values of *k* such that $1 \le k\pi \le 500$; that is, $0.318 \le k \le 159.15$. So $k = 1, 2, 3, \ldots, 159$, which is 159 values of *k*; that is, 159 values of *x*.