Forty-Eighth Annual

NEAML New England Championship Math Meet

Friday, May 20, 2022

Conducted by

THE NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

SOLUTIONS

NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

FORTY-EIGHTH NEW ENGLAND PLAYOFFS

Friday, May 20, 2022

SOLUTIONS

Round 1 — Arithmetic and Number Theory

1. Write as a reduced proper fraction:

$$1+\cfrac{1+\cfrac{1+\cfrac{1+\cfrac{1+\frac{1}{2}}{3}}{4}}{5}}{1+\cfrac{5}{1+\cfrac{4}{1+\cfrac{3}{1+\frac{2}{1}}}}}$$

   Answer: $\boxed{\frac{473}{320}}$

   Solution: (This question is NEAML 1973, R1Q3.) Starting with the top of the fraction, we have $1+\frac{1}{2}=\frac{3}{2}$; $1+\frac{\mathbf{3/2}}{3}=\frac{3}{2}$; $1+\frac{\mathbf{3/2}}{4}=\frac{11}{8}$; $1+\frac{\mathbf{11/8}}{5}=\frac{51}{40}$.

   Continuing with the bottom of the fraction, we have $1+\frac{2}{1}=3$; $1+\frac{3}{\mathbf{3}}=2$; $1+\frac{4}{\mathbf{2}}=3$; $1+\frac{5}{\mathbf{3}}=\frac{8}{3}$.

   Putting these together, our fraction is $1+\frac{51}{40}\div\frac{8}{3}=1+\frac{51}{40}\cdot\frac{3}{8}=1+\frac{153}{320}=\boxed{\frac{473}{320}.}$

2. In Slovinkia, 90% of the people drink tea, 80% drink coffee, 70% drink orange juice, and 60% drink grape juice. No one in Slovinkia drinks all four of these beverages. What percent of the people in Slovinkia drink at least one of the two juices?

   Answer: $\boxed{100\%}$

   Solution: (This question is NEAML 1980, R1Q3.) For convenience, assume there are 100 people in Slovinkia. Let A={people who drink tea}, B={people who drink coffee}, C={people who drink orange juice}, D={people who drink grape juice}. We are given $|A\cap B\cap C\cap D|=0$, so every person *does not drink* at least one thing. Hence $|A^C|+|B^C|+|C^C|+|D^C|\geq 100$, because each person is contained in at least one, possibly more, of the four complements.

   But we are given that $|A^C|=10$, $|B^C|=20$, $|C^C|=30$, and $|D^C|=40$, so $|A^C|+|B^C|+|C^C|+|D^C|=100$ exactly. Therefore no person is in more than one of the four complements, but every person is in at least one, so each person is in exactly one of the four complements. This means every person drinks exactly three of the drinks, and does not drink exactly one other. So if someone does not drink grape juice, they must drink orange juice, and vice-versa. Therefore $\boxed{100\%}$ of the population drinks at least one juice.

3. A girl has a number of books to pack into parcels. If she packs 2, 3, 4, 5, or 6 books per parcel, there is always one book left unpacked. If she packs 7 books per parcel, none are left over. What is the minimum number of books she has to pack?

Answer: $\boxed{301}$

Solution: (This question is NEAML 1974, T1.) The number of books the girl has to pack is one more than a multiple of 2, 3, 4, 5, and 6; therefore, one more than a common multiple of these five numbers. The least common multiple of these numbers is $2^2 \cdot 3 \cdot 5 = 60$. So the girl has a number of books which is in the set $\{1, 61, 121, 181, 241, 301, 361, \ldots\}$. We need to to find the smallest number in this set which is a multiple of 7.

A little-known divisibility rule for 7 says that an integer is divisible by 7 iff you can slice off the last digit, double it, subtract it from the remaining number, and get another multiple of 7. For example, 315 is a multiple of 7 because when you slice off the 5, double it, and subtract from what's left, you get $\mathbf{31} - 2 \cdot \mathbf{5} = 21$, and 21 is a multiple of 7.

Applying this rule to successive terms in the set above, we get $\{-2, 4, 10, 16, 22, 28, 34, \cdots\}$. The first multiple of 7 in this sequence is 28, which came from 301.

So the minimum number of books the girl has to pack is $\boxed{301.}$

Round 2 — Algebra I

1. A picture of a Vermont covered bridge was 3 inches longer than it was wide. It was mounted on white poster paper, leaving a 2-inch wide border on all sides of the picture. If the area of the sheet of white poster paper is 76 square inches greater than the area of the picture, find the length of the picture in inches.

   Answer: $\boxed{9 \textit{ or } 9 \text{ inches}}$

   Solution: (This question is NEAML 1973, R2Q3.) Let the width of the picture be x. Then the area of the picture is $(x)(x+3)$ while the area of the poster paper is $(x+4)(x+7)$.

$$(x+4)(x+7) - (x)(x+3) = 76$$
$$(x^2 + 11x + 28) - (x^2 + 3x) = 76$$
$$8x + 28 = 76$$
$$8x = 48$$
$$x = 6$$

   Therefore, the length of the original picture is $6 + 3 = \boxed{9 \text{ inches.}}$

2. Find all ordered pairs of real numbers (x, y), with $x \geq y$, such that

$$2x^2 + 2y^2 + x + y = 6$$

   and

$$4xy + x + y = -2.$$

   Answer: $\boxed{(0, -2) \text{ and } (3/2, -1/2) \text{ (either order)}}$

   Solution: (This question is NEAML 1977, R2Q3.) From the second equation we have that

$$x + y = -4xy - 2.$$

   Plug this into the first equation to get

$$2x^2 + 2y^2 - 4xy - 2 = 6$$
$$2x^2 + 2y^2 - 4xy = 8$$
$$x^2 + y^2 - 2xy = 4$$
$$(x - y)^2 = 2^2$$

   So $x - y = \pm 2$. But $x \geq y$, so $x - y = 2$. Plug $y = x - 2$ into the second equation:

$$4x(x-2) + x + (x-2) = -2$$
$$4x^2 - 8x + 2x - 2 = -2$$
$$4x^2 - 6x = 0$$
$$2x(2x - 3) = 0$$

   So either $x = 0$ or $x = 3/2$; and because $y = x - 2$, we have that $y = -2$ or $y = -1/2$, respectively. Thus our ordered pairs are $\boxed{(0, -2) \text{ and } (3/2, -1/2).}$

3. If each of four numbers is added to the average of the other three, the respective sums are 27, 29, 33, and 37. Find the largest of the four numbers.

Answer: $\boxed{24}$

Solution: (This question is NEAML 1979, R2Q3.) Call the four numbers a, b, c, and d. We are thus told that

$$\frac{b+c+d}{3}+a=27$$
$$\frac{a+c+d}{3}+b=29$$
$$\frac{a+b+d}{3}+c=33$$
$$\frac{a+b+c}{3}+d=37.$$

Adding these four symmetric equations, we get

$$\frac{3a+3b+3c+3d}{3}+(a+b+c+d)=126$$
$$a+b+c+d=63.$$

Tripling the fourth equation, we get

$$a+b+c+3d=111.$$

(the largest right hand side we can get), and subtracting the previous equation we get

$$2d=111-63=48.$$

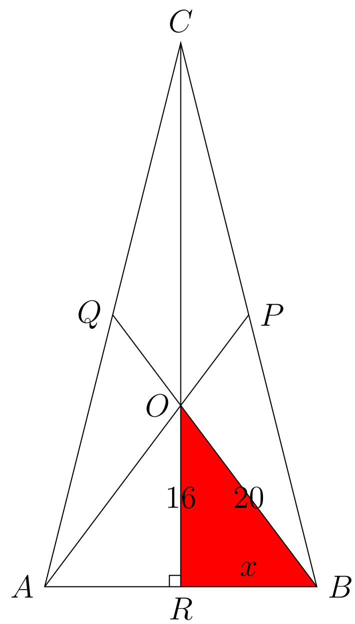
So $\boxed{d=24.}$ (Using similar steps we can deduce that $a=9$, $b=12$, and $c=18$.)

Round 3 — Geometry

1. The lengths of the medians of a triangle are 30, 30, and 48. Find the length of the shortest side of this triangle.

   Answer: $\boxed{24}$

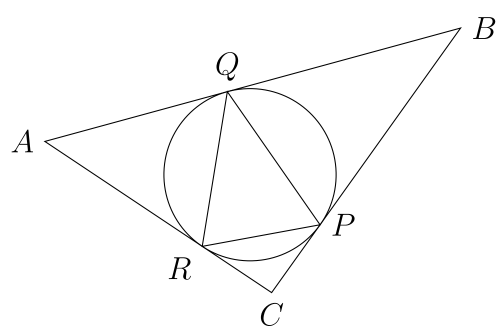
   Solution: (This question is NEAML 1979, R3Q1.) Drawing a diagram:



   Recall that the medians of a triangle intersect in a point which divides each in a 2 : 1 ratio, the larger piece near the vertex and the smaller near the midpoint of the opposite side.

   The equal medians $\overline{AP}$ and $\overline{BQ}$ tell us the triangle is isosceles ($\triangle QOA \cong \triangle POB$ by SAS, so $QA = PB$, so $CA = CB$), so the median $\overline{CR}$ of length 48 is also an altitude, perpendicular to the base of the isosceles triangle. This means we have an x–16–20 *right* triangle, where x is half the length of the base. So $x = 12$ and the base is $\boxed{24}$. (The Pythagorean Theorem tells us that the legs each have length $\sqrt{12^2 + 48^2} = 12\sqrt{1^2 + 4^2} = 12\sqrt{17}$.)

2. If triangle ABC circumscribes a circle and the points of tangency are joined to form triangle PQR as shown, what is the measure of $\angle QPR$ if the measure of $\angle A$ is x degrees? Give your answer in terms of x.



   Answer: $\boxed{\frac{180-x}{2} \text{ or equivalent}}$

Solution: (This question is NEAML 1973, R3Q2.) By a standard circle theorem,

$$m\angle A = \frac{1}{2}\left(m\widehat{QPR} - m\widehat{QR}\right),$$

but these two arcs sum to 360°, so

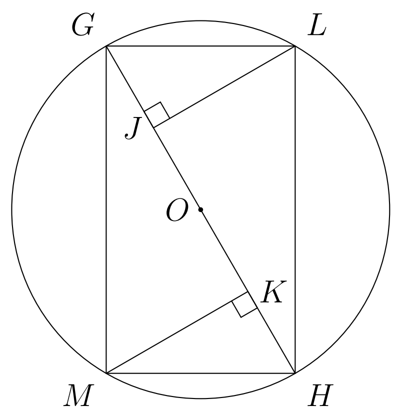
$$m\angle A = \frac{1}{2}\left(\left(360^\circ - m\widehat{QR}\right) - m\widehat{QR}\right)$$

$$m\angle A = 180^\circ - m\widehat{QR},$$

which is to say, $m\angle A$ and $m\widehat{QR}$ sum to 180°. (That circle theorem might be in your repertoire.) Since $m\angle A = x$, $m\widehat{QR} = 180 - x$.

By the inscribed angle theorem, $m\angle QPR = \frac{1}{2}m\widehat{QR}$, which equals $\boxed{\frac{180-x}{2}}$.

3. To cut the stiffest possible rectangular beam of cross-section $GMHL$ from a cylidrical log, a sawyer divides the diameter $\overline{GH}$ at J, O, and K so that $GJ = JO = OK = KH$. He then constructs $\overline{JL}$ and $\overline{KM}$ perpendicular to $\overline{GH}$. He then draws $\overline{GM}$, $\overline{MH}$, $\overline{HL}$, and $\overline{LG}$ and cuts through them. If the diameter $\overline{GH}$ of the cylindrical log is 2 feet long, find the area of rectangle $GMHL$ in square feet.



Answer: $\boxed{\sqrt{3} \textit{ or } \sqrt{3} \text{ square feet}}$

Solution: (This question is a modification of NEAML 1974, T3.) By the inscribed angle theorem, $\triangle GLH$ is right. Therefore JL, the altitude to the hypotenuse, is the geometric mean of GJ and JH. So $JL = \sqrt{\frac{1}{2} \cdot \frac{3}{2}} = \frac{\sqrt{3}}{2}$. By similar triangles,

$$\frac{GL}{LH} = \frac{GJ}{JL}$$

$$\frac{GL}{LH} = \frac{1/2}{\sqrt{3}/2}$$

$$\frac{GL}{LH} = \frac{1}{\sqrt{3}}$$

By the Pythagorean theorem, $GL = \sqrt{(\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} = 1$.

So $GL = 1$ and $LH = \sqrt{3}$, giving us the area $\sqrt{3} \cdot 1 = \boxed{\sqrt{3}}$

Round 4 — Algebra II

1. If $f(x) = ax^5 + bx^3 + cx + 3$, and if $f(9) = -7$, find $f(-9)$.

   Answer: $\boxed{13}$

   Solution: (This question is NEAML 1975, R4Q2.)

$$f(-9)$$

$$\begin{gathered} a(-9)^5 + b(-9)^3 + c(-9) + 3 \\ -a(9)^5 - b(9)^3 - c(9) - 3 + 6 \\ -(a(9)^5 + b(9)^3 + c(9) + 3) + 6 \\ -f(9) + 6 \\ 7 + 6 \\ \boxed{13} \end{gathered}$$

2. Find x if 3^{3-x}, 3^{-x}, and $3^{\sqrt{9-2x}}$ are in geometric progression.

   Answer: $\boxed{-8}$

   Solution: (This question is NEAML 1975, T4.) If three powers of 3 are in geometric progression, their exponents must be in arithmetic progression, with a common difference. Therefore,

$$(-x) - (3 - x) = \sqrt{9 - 2x} - (-x)$$

$$\begin{gathered} -3 = \sqrt{9 - 2x} + x \\ -x - 3 = \sqrt{9 - 2x} \end{gathered}$$

   We square both sides, potentially introducing extraneous solutions:

$$x^2 + 6x + 9 = 9 - 2x$$

$$\begin{gathered} x^2 + 8x = 0 \\ x(x + 8) = 0. \end{gathered}$$

   So $x = 0$ or $x = -8$, but $x = 0$ is an extraneous solution (since $-0 - 3 = \sqrt{9 - 2 \cdot 0}$ gives $-3 = \sqrt{9}$). So the answer is $\boxed{-8.}$

3. Find the integer k for which

$$\sqrt[3]{1-12\sqrt[3]{7}+6\sqrt[3]{49}}+\sqrt[3]{7}=\sqrt[3]{k}.$$

Answer: $\boxed{8}$

Solution: (This question is NEAML 1978, R4Q3.) Move the $\sqrt[3]{7}$ to the right side, cube both sides, then collect terms on one side:

$$\sqrt[3]{1-12\sqrt[3]{7}+6\sqrt[3]{49}}=\sqrt[3]{k}-\sqrt[3]{7}$$

$$1-12\sqrt[3]{7}+6\sqrt[3]{49}=k-7-3\sqrt[3]{7k^2}+3\sqrt[3]{49k}$$

$$(8-k)-12\sqrt[3]{7}+3\sqrt[3]{7k^2}+6\sqrt[3]{49}-3\sqrt[3]{49k}=0$$

Factoring, we get

$$(8-k)-3\sqrt[3]{7}(4-\sqrt[3]{k^2})+3\sqrt[3]{49}(2-\sqrt[3]{k})=0$$

This equation would be true if $8-k$, $4-\sqrt[3]{k^2}$, and $2-\sqrt[3]{k}$ were simultaneously equal to zero for some value of k; and, in fact, $\boxed{k=8}$ does the trick.

Alternate solution: Perhaps more eloquently, one could also do the following:

$$\sqrt[3]{1-12\sqrt[3]{7}+6\sqrt[3]{49}}$$

$$\sqrt[3]{8-12\sqrt[3]{7}+6\sqrt[3]{49}-7}$$

$$\sqrt[3]{8-3\cdot 2^2\sqrt[3]{7}+3\cdot 2\sqrt[3]{49}-7}$$

$$\sqrt[3]{2^3-3\cdot 2^2\sqrt[3]{7}+3\cdot 2\sqrt[3]{7^2}-\sqrt[3]{7^3}}$$

$$\sqrt[3]{2^3-3\cdot 2^2\sqrt[3]{7}+3\cdot 2(\sqrt[3]{7})^2-(\sqrt[3]{7})^3}$$

$$\sqrt[3]{(2-\sqrt[3]{7})^3}$$

$$2-\sqrt[3]{7},$$

in which case $\sqrt[3]{k}=2$, and therefore $\boxed{k=8.}$

Round 5 — Analytic Geometry

1. The graph of a quadratic function $f(x) = a^2 + bx + c$ is a parabola which passes through the points with coordinates $(1, 1)$, $(2, 4)$, and $(5, 1)$. Find the coordinates of the vertex of this parabola.

   Answer: $\boxed{(3, 5)}$

   Solution: (This question is NEAML 1976, R5Q1.) As the graph of a quadratic function, this is a parabola with a vertical axis of symmetry. Because it passes through $(1, 1)$ and $(5, 1)$, its axis of symmetry is $x = 3$. Because the vertex lies on the axis of symmetry, all that remains is to find the y-coordinate of the vertex.

   Shifting the parabola down one unit, the function would pass through $(1, 0)$ and $(5, 0)$, so its equation is $f(x) - 1 = a(x - 1)(x - 5)$. Solving for $f(x)$, the quadratic function can be written as $f(x) = a(x - 1)(x - 5) + 1$.

   Plug in $x = 2$ to get $f(2) = a(2 - 1)(2 - 5) + 1 = 1 - 3a = 4$, so $a = -1$ and the quadratic function is $f(x) = -(x - 1)(x - 5) + 1$.

   Plug in $x = 3$ to get $f(3) = -(3 - 1)(3 - 5) + 1 = 5$.

   Therefore the coordinates of the vertex are $\boxed{(3, 5).}$

2. Find all values of m for which the line $y = mx + 5$ will *not* intersect the circle $x^2 + y^2 = 9$ in the real plane. Give your answer in interval notation.

   Answer: $\boxed{\left(-\frac{4}{3}, \frac{4}{3}\right)}$

   Solution: (This question is NEAML 1979, R5Q1.) Solve the equations simultaneously by substituting $mx + 5$ for y in the equation of the circle.

   $$x^2 + (mx + 5)^2 = 9$$

   $$x^2 + m^2x^2 + 10mx + 25 = 9$$
   $$(1 + m^2)x^2 + (10m)x + 16 = 0$$

   This quadratic has no solutions when its discriminant is less than zero:

   $$(10m)^2 - 4(1 + m^2)(16) < 0$$

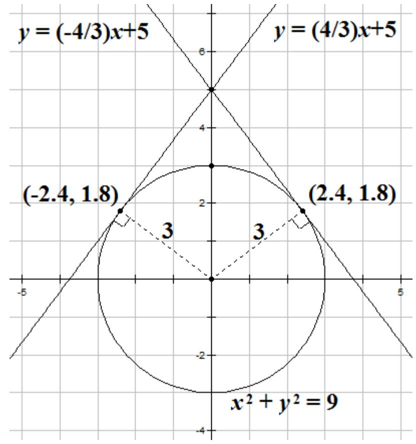
   $$100m^2 - 64 - 64m^2 < 0$$
   $$36m^2 - 64 < 0$$
   $$m^2 < \frac{64}{36} = \frac{16}{9}$$

   Therefore

   $$-\frac{4}{3} < m < \frac{4}{3}.$$

   Written in interval notation, the line does not intersect the circle when $m \in \boxed{\left(-\frac{4}{3}, \frac{4}{3}\right).}$

   Here is a graph showing the borderline cases:



3. A *latus rectum* of an ellipse is a chord of the ellipse which is perpendicular to the major axis and which passes through a focus. If the equation of a conic section is

$$x^2 + 4y^2 - 6x + 16y + 9 = 0,$$

find the area of the rectangle which has its two latus recta as opposite sides.

Answer: $\boxed{8\sqrt{3}}$

Solution: (This question is NEAML 1973 R5Q3.) Putting the conic into standard form, we have

$$x^2 + 4y^2 - 6x + 16y + 9 = 0$$

$$\begin{aligned} x^2 - 6x \quad\quad + 4(y^2 + 4y \quad\quad) &= -9 \\ x^2 - 6x + 9 + 4(y^2 + 4y + 4) &= -9 + 9 + 16 \\ (x-3)^2 + 4(y+2)^2 &= 16 \\ \frac{(x-3)^2}{4^2} + \frac{(y+2)^2}{2^2} &= 1. \end{aligned}$$

Because area is unchanged after translation, consider the ellipse

$$\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1.$$

The foci are located on the major axis at $(\pm\sqrt{4^2 - 2^2}, 0)$, so the width of the required rectangle is $2\sqrt{12} = 4\sqrt{3}$. Plugging $x = \sqrt{12}$ into the equation, we have

$$\frac{12}{16} + \frac{y^2}{4} = 1$$

$$\frac{3}{4} + \frac{y^2}{4} = 1$$

$$\frac{y^2}{4} = \frac{1}{4}$$

So $y = \pm 1$, and the height of the required rectangle is 2. Therefore, the area of the required rectangle is $2 \cdot 4\sqrt{3} = \boxed{8\sqrt{3}}$.

Round 6 — Trigonometry and Complex Numbers

1. Given that a and b are real numbers, $z = a + bi$, and $\overline{z}$ is the complex conjugate of z, find the value of $a^2 + b^2$ if $(z + \overline{z})z = 2 + 4i$.

   Answer: $\boxed{5}$

   Solution: (This question is NEAML 1980, R6Q1.) Plugging in $z = a + bi$ and $\overline{b} = a - bi$, we get

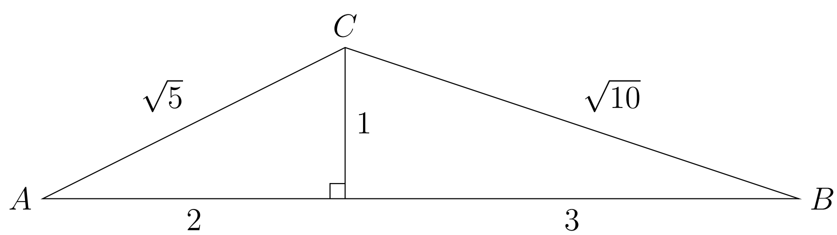
$$((a + bi) + (a - bi))(a + bi) = 2 + 4i$$
$$(2a)(a + bi) = 2 + 4i$$
$$(2a^2 + 2abi) = 2 + 4i$$

   Setting real and imaginary terms equal, we have $2a^2 = 2$, whence $a = \pm 1$, and $2ab = 4$, whence $b = \pm 2$ (where a and b are both positive or both negative). The question seeks $a^2 + b^2$ which is, in either case, $\boxed{5.}$

2. In $\triangle ABC$, $\tan A = 1/2$ and $\tan B = 1/3$. Find the degree measure of angle C.

   Answer: $\boxed{135°}$

   Solution: (This question is NEAML 1976, R6Q3.) Drawing a quick diagram:



   The tangent-of-a-sum formula states that

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}.$$

   So, in this case, we have

$$\tan(A + B) = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \cdot \frac{1}{3}}$$
$$\tan(A + B) = \frac{\frac{5}{6}}{1 - \frac{1}{6}}$$
$$\tan(A + B) = 1.$$

   Now,

$$\tan C = \tan(180° - (A + B))$$
$$\tan C = -\tan(A + B)$$
$$\tan C = -1.$$

   Because $\angle C$ must have a degree measure between 0 and 180°, $m\angle C = \boxed{135°.}$

   Alternate solution: Setting the area of the triangle equal to itself, we have

$$\frac{1}{2} \cdot 5 \cdot 1 = \frac{1}{2}\sqrt{5}\sqrt{10} \sin C$$

$$5 = 5\sqrt{2}\sin C$$
$$\sin C = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2},$$

so angle C measures 45° or 135°. But angle C is clearly obtuse, so the answer is $\boxed{135°.}$

Alternate solution: Use the Law of Cosines:

$$5^2 = 5 + 10 - 2\sqrt{50}\cos C$$
$$25 = 15 - 10\sqrt{2}\cos C$$
$$10 = -10\sqrt{2}\cos C$$
$$\cos C = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

which gives $\boxed{135°}$ for the measure of angle C.

3. Find the value(s) of θ, $0 \le \theta \le \pi$, such that

$$5\tan\theta - \sqrt{2}\sec\theta + 3 = 0.$$

Give your answer in radians. If necessary, express your answer(s) in terms of an inverse trigonometric function.

Answer: $\boxed{3\pi/4}$

Solution: (This question is a modification of NEAML 1982, R6Q3.) Rather than converting everything to sines and cosines, recall the Pythagorean identity $\sec^2\theta = \tan^2\theta + 1$.

$$5\tan\theta - \sqrt{2}\sec\theta + 3 = 0$$
$$5\tan\theta - \sqrt{2}\sqrt{\tan^2\theta + 1} + 3 = 0$$
$$5\tan\theta + 3 = \sqrt{2(\tan^2\theta + 1)}$$

Square both sides, possibly introducing extraneous solutions:

$$25\tan^2\theta + 30\tan\theta + 9 = 2\tan^2\theta + 2$$
$$23\tan^2\theta + 30\tan\theta + 7 = 0$$
$$(23\tan\theta + 7)(\tan\theta + 1) = 0$$

So $\tan\theta = -1$ or $\tan\theta = -7/23$. We can confirm that $\tan\theta = -1$ is a valid solution, because this is in the second quadrant, where $\sec\theta$ is $-\sqrt{2}$. So $5(-1) - \sqrt{2}(-\sqrt{2}) + 3 = -5 + 2 + 3 = 0$, as required.

We can also see that $\tan\theta = -7/23$ is not a valid solution, because in the second quadrant, the value of $\sec\theta$ is negative. Rewrite the original equation as

$$5\tan\theta + 3 = \sqrt{2}\sec\theta.$$

On the left hand side, we have $5(-\frac{7}{23}) + 3 = -\frac{35}{23} + \frac{69}{23} = \frac{34}{23}$, which is positive; and on the right hand side, we have $\sqrt{2}\sec\theta$, which is negative, a contradiction.

Thus the only answer occurs when $\tan\theta = -1$, which for $0 \le \theta \le \pi$ occurs when $\theta = \boxed{3\pi/4.}$

Team Round

1. Find the least positive remainder obtained when 30! is divided by 5^8.

Answer: $\boxed{312{,}500}$

Solution: (This question is NEAML 1979, R1Q3.) We are seeking R in the equation

$$30! = 5^8 Q + R.$$

Notice that 30! contains 5^7, one factor of 5 from $\{5, 10, 15, 20, 30\}$ and two from 25. 5^8 also contains 5^7, obviously, so we know that R contains 5^7 (because the difference of two multiples of 5^7 is itself a multiple of 5^7.

Therefore, divide the equation by 5^7:

$$\frac{30!}{5^7} = 5Q + \frac{R}{5^7}.$$

(Despite the fractions, all three of these numbers are integers.)

Thus we have

$$\frac{R}{5^7} \equiv \frac{30!}{5^7} \pmod 5.$$

Expanding this, we have

$$\frac{R}{5^7} \equiv 6{\cdot}29{\cdot}28{\cdot}27{\cdot}26{\cdot}1{\cdot}24{\cdot}23{\cdot}22{\cdot}21{\cdot}4{\cdot}19{\cdot}18{\cdot}17{\cdot}16{\cdot}3{\cdot}14{\cdot}13{\cdot}12{\cdot}11{\cdot}2{\cdot}9{\cdot}8{\cdot}7{\cdot}6{\cdot}1{\cdot}4{\cdot}3{\cdot}2{\cdot}1 \pmod 5.$$

Consider each group of four numbers between successive multiples of 5. Each group is equivalent to $4 \cdot 3 \cdot 2 \cdot 1 \equiv 24 \equiv -1 \pmod 5$.

$$\frac{R}{5^7} \equiv 6 \cdot \underbrace{29 \cdot 28 \cdot 27 \cdot 26}_{-1} \cdot \underbrace{24 \cdot 23 \cdot 22 \cdot 21}_{-1} \cdot 4 \cdot \underbrace{19 \cdot 18 \cdot 17 \cdot 16}_{-1} \cdot 3 \cdot \underbrace{14 \cdot 13 \cdot 12 \cdot 11}_{-1} \cdot 2 \cdot \underbrace{9 \cdot 8 \cdot 7 \cdot 6}_{-1} \cdot \underbrace{4 \cdot 3 \cdot 2 \cdot 1}_{-1}.$$

So

$$\frac{R}{5^7} \equiv 6 \cdot 4 \cdot 3 \cdot 2 \cdot (-1)^6 \pmod 5$$

$$\frac{R}{5^7} \equiv 24 \equiv 4 \pmod 5$$

Therefore $\frac{R}{5^7} = 4$, so $R = 5^7 \cdot 4 = 5^5 \cdot 100 = \boxed{312{,}500.}$

2. Find all real values of x which satisfy

$$\frac{x-1}{x} \le \frac{x}{x-1}.$$

Give your answer in interval notation

Answer: $\boxed{(0, \tfrac{1}{2}] \cup (1, \infty).\text{ A correct answer must contain the union symbol or the word "or."}}$

Solution: (This question is NEAML 1978, R2Q3.) We want to know when the graph of the function on the right is above the graph of the function on the left. Begin by asking when the relative positions of these graphs can change: only at a discontinuity or at an intersection.

Discontinuities occur at $x = 0$ and $x = 1$.

Intersections occur where

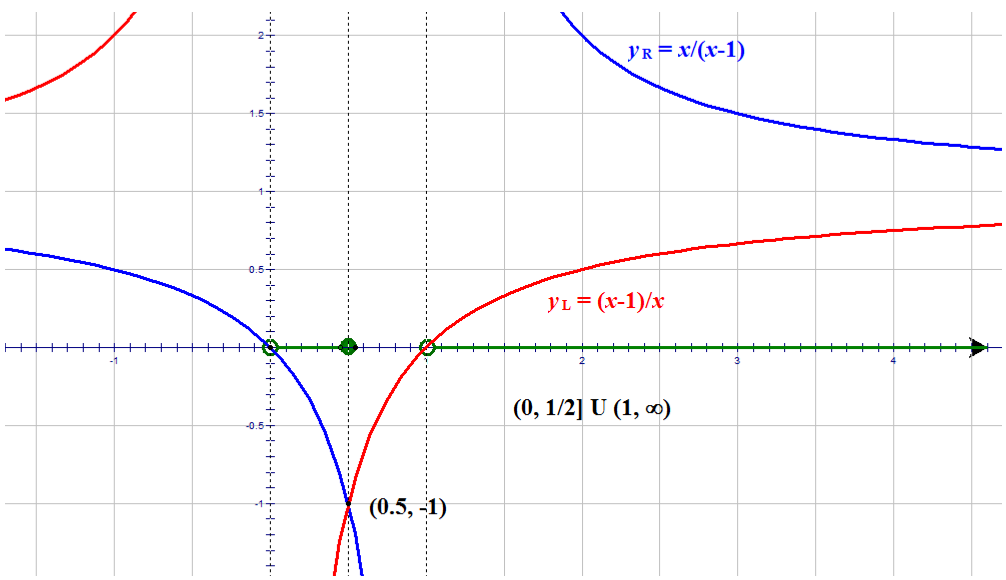
$$\frac{x-1}{x} = \frac{x}{x-1}$$
$$(x-1)^2 = x^2$$
$$x^2 - 2x + 1 = x^2$$
$$2x = 1$$
$$x = \frac{1}{2}.$$

The relative positions of the two graphs can change only at 0, $\frac{1}{2}$, and 1. All that remains is to test the intervals between these numbers. For very large x, the inequality is true (e.g., $99/100 \le 100/99$). For negative numbers, the inequality is false (e.g., $\frac{-100}{-99} \not\le \frac{-99}{-100}$ because $\frac{100}{99} \not\le \frac{99}{100}$). When $x = \frac{1}{4}$, the inequalty says $-3 \le -\frac{1}{3}$, which is true; and when $x = \frac{3}{4}$, the inequality says $-\frac{1}{3} \le -3$, which is false.

Additionally, because we allow equality, 1/2 is also in the solution set.

Putting this together, we have $\boxed{(0, \frac{1}{2}] \cup (1, \infty).}$

Alternate solution: Consider the two functions separately. $f(x) = \frac{x-1}{x} = 1 - \frac{1}{x}$ is a rectangular hyperbola flipped vertically and shifted up one unit. $g(x) = \frac{x}{x-1} = \frac{(x-1)+1}{x-1} = 1 + \frac{1}{x-1}$, a rectangular hyperbola shifted one unit right and one unit up. A sketch of these two functions looks like this:

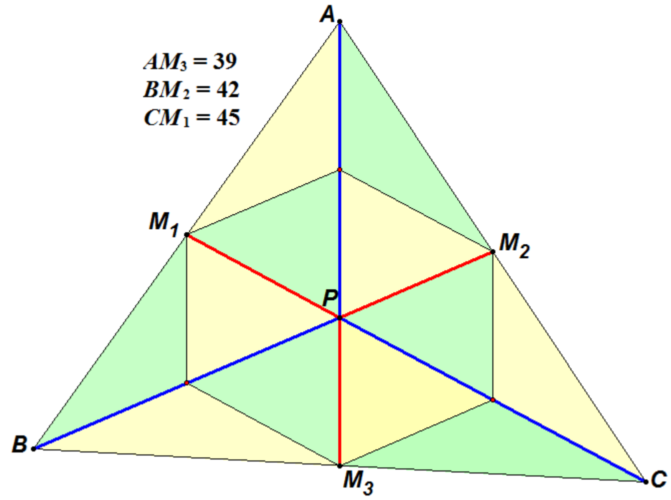


We want to know when the blue graph ($f(x)$) is above the red graph. After working out the intersection at $x = \frac{1}{2}$, one can find the same solution.

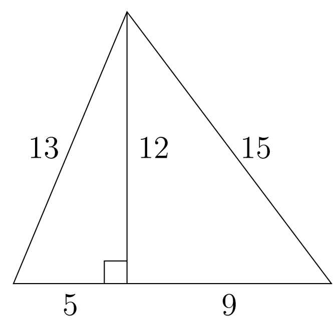
3. Find the area of a triangle in which the lengths of the three medians are 39, 42, and 45

   Answer: $\boxed{1008}$

   Solution: (This question is NEAML 1980, T3.) The three medians intersect at the centroid of the triangle, which divides each into a 2 : 1 ratio. Bisecting the longer piece of each median allows us to create twelve smaller triangles with equal area, as shown:



Notably, each of the six triangles nearest the centroid is a 13–14–15 triangle, which famously is a 5–12–13 and 9–12–15 triangle joined along the side of length 12:



So each of the twelve triangles has area $\frac{1}{2} \cdot 14 \cdot 12 = 84$. Since the given triangle is composed of 12 triangles with this area, the area of the given triangle is $84 \cdot 12 = \boxed{1008.}$

Alternate solution: Use Heron's formula to find the area of each 13-14-15 triangle:

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

$$A = \sqrt{21(8)(7)(6)}$$

$$A = \sqrt{(7 \cdot 3)(4 \cdot 2)(7)(3 \cdot 2)}$$

$$A = 7 \cdot 3 \cdot 4$$

$$A = 84,$$

then continue as above.

4. Let $f(x) = ax^2 + bx + c$ denote a quadratic function with discriminant 4. $f(6) = 0$ and $f(0) = 6$. There are two such functions f. Express the coefficients a, b, and c of each function as an ordered triple (a, b, c).

Answer: $\boxed{\left(\frac{1}{2}, -4, 6\right) \text{ and } \left(-\frac{1}{6}, 0, 6\right) \text{ (either order)}}$

Solution: (This question is a modification of NEAML 1973, R4Q3.) Assume the function is $f(x) = ax^2 + bx + c$. Because $f(0) = 6$, we immediately have $c = 6$.

Using Vieta's formulas, we know that the product of the roots is $\frac{6}{a}$; but we also know the product of the roots is $6r_2$, because one of the roots is 6. So $\frac{6}{a} = 6r_2$, whence $r_2 = \frac{1}{a}$.

Therefore the function has the form $f(x) = a(x-6)(x-\frac{1}{a}) = (x-6)(ax-1) = ax^2 - (6a+1)x + 6$. Plugging this into the discriminant formula, we have

$$(6a+1)^2 - 4(a)(6) = 4$$

$$36a^2 + 12a + 1 - 24a = 4$$

$$36a^2 - 12a - 3 = 0$$

$$(6a-3)(6a+1) = 0$$

$$a = \frac{1}{2} \qquad \text{or} \qquad a = -\frac{1}{6}$$

If $a = \frac{1}{2}$, the function is $f(x) = \frac{1}{2}x^2 - 4x + 6$ and if $a = -\frac{1}{6}$, the function is $f(x) = -\frac{1}{6}x^2 + 6$. The two ordered triples, then, are $\boxed{\left(\frac{1}{2}, -4, 6\right) \text{ and } \left(-\frac{1}{6}, 0, 6\right),}$ respectively.

5. A ship is lying somewhere on a line $250\sqrt{2}$ kilometers from and parallel to a (linear) shore. Two LORAN posts established at points P and Q on the shore emit signals which are received by the ship. From the time lapse between the signals it was possible to calculate that the ship is $100\sqrt{6}$ kilometers closer to Q than to P. Taking 1 unit = 50 kilometers, a set of coordinate axes was drawn with the x-axis representing the shore. Wth respect to these axes, the posts were located at $P(2,0)$ and $Q(10,0)$. With respect to these same axes, determine the coordinates of the position of the ship.

   Answer: $\boxed{(12, 5\sqrt{2})}$

   Solution: (This question is NEAML 1975, R5Q3.) The set of all points such that the differences of the distances to two fixed points (foci) is a constant is, of course, a hyperbola. One way to approach this problem is to find the equation of the hyperbola. In a hyperbola, we have $a^2 + b^2 = c^2$, where c is the distance from the center to the foci. Assuming the foci have the same y-coordinate, as is true in this case, then a is half the transverse axis (the distance between the vertices) and b is half the conjugate axis (the height of the "guide box" used to draw the asymptotes with slopes $\pm\frac{b}{a}$).

   So in this case, the center is at $(6,0)$ and $c = 4$. The fixed difference between the two distances always equals the length of the transverse axis (similar to how in an ellipse the fixed sum of the two distances always equals the length of the major axis), so we have $2a = 2\sqrt{6}$ or $a = \sqrt{6}$. We can then calculate that $b^2 = 4^2 - (\sqrt{6})^2 = 10$, so $b = \sqrt{10}$. Then we know the hyperbola in question is

$$\frac{(x-6)^2}{6} - \frac{y^2}{10} = 1$$

   The question says that the y-coordinate of the ship is $5\sqrt{2}$, so plug that in to find x:

$$\frac{(x-6)^2}{6} - \frac{(5\sqrt{2})^2}{10} = 1$$

$$\frac{(x-6)^2}{6} - \frac{50}{10} = 1$$

$$\frac{(x-6)^2}{6} - 5 = 1$$

$$\frac{(x-6)^2}{6} = 6$$

$$(x-6)^2 = 6^2$$

$$x - 6 = \pm 6$$

So $x = 0$ or $x = 12$, but because the ship is closer the station at $(10, 0)$, the x-value must be 12. So the coordinates are $\boxed{(12, 5\sqrt{2}).}$

Alternate solution: Model the situation using the distance formula:

$$\sqrt{(x-2)^2 + (5\sqrt{2})^2} - 2\sqrt{6} = \sqrt{(x-10)^2 + (5\sqrt{2})^2}$$

$$\sqrt{x^2 - 4x + 54} - 2\sqrt{6} = \sqrt{x^2 - 20x + 150}$$

Square both sides, possibly introducing an extraneous root:

$$x^2 - 4x + 54 + 24 - 4\sqrt{6}\sqrt{x^2 - 4x + 54} = x^2 - 20x + 150$$

Isolate the square root:

$$-4x + 54 + 24 - 4\sqrt{6}\sqrt{x^2 - 4x + 54} = -20x + 150$$

$$16x - 72 = 4\sqrt{6}\sqrt{x^2 - 4x + 54}$$

$$4x - 18 = \sqrt{6}\sqrt{x^2 - 4x + 54}$$

Square both sides again:

$$16x^2 - 144x + 324 = 6(x^2 - 4x + 54)$$

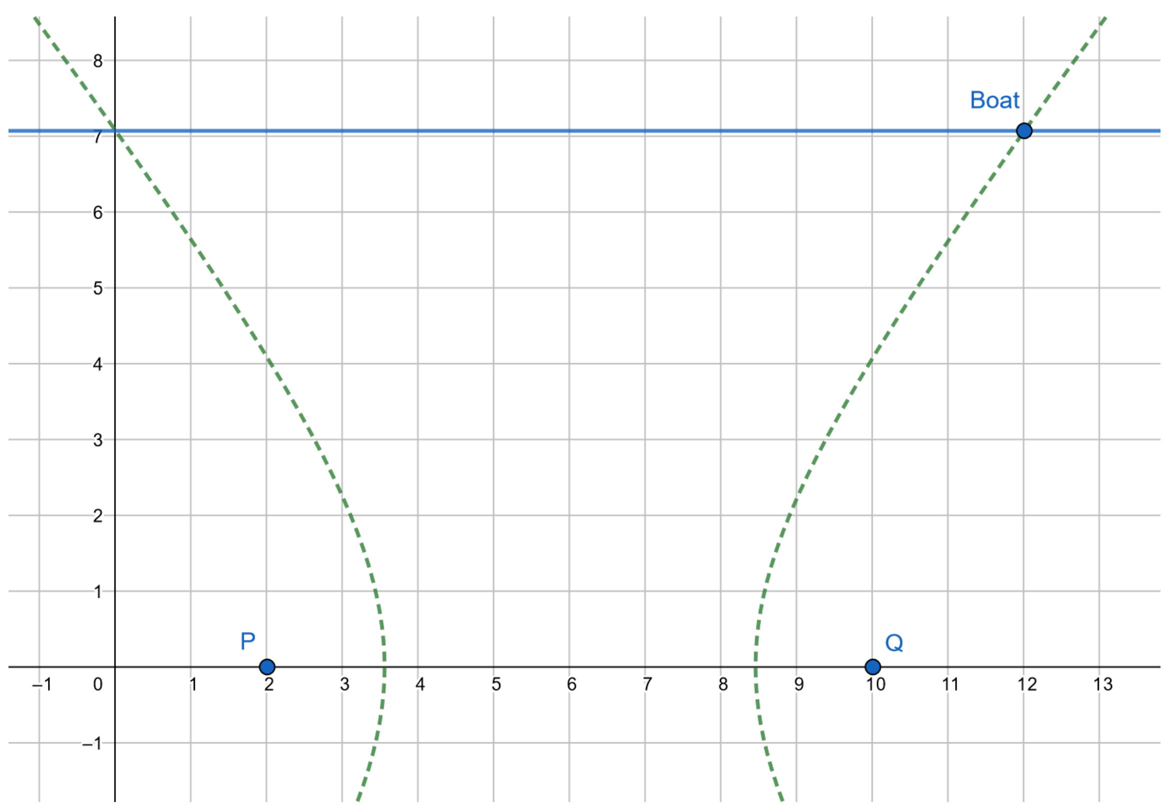
$$16x^2 - 144x + 324 = 6x^2 - 24x + 324$$

$$10x^2 - 120x = 0$$

$$x^2 - 12x = 0$$

So $x = 0$ or $x = 12$. $x = 0$ is extraneous, because the point $(0, 5\sqrt{2})$ is closer to P than to Q. So the answer is $x = 12$, and the desired coordinates are $\boxed{(12, 5\sqrt{2}).}$

Here is a graph showing the LORAN hyperbola.



6. Determine the simplified numerical value of

$$(\sin 10^\circ)(\sin 30^\circ)(\sin 50^\circ)(\sin 70^\circ).$$

Answer: $\boxed{1/16}$

Solution: (This question is NEAML 1979, T6.) Let's start by deriving two product-to-sum identities. Because

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

and

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta,$$

we can subtract these identities to get the identity

$$\cos(\alpha-\beta)-\cos(\alpha+\beta)=2\sin\alpha\sin\beta,$$

or

$$\sin\alpha\sin\beta=\frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta)).$$

Also, because

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

and

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta,$$

we can add these identities to get

$$\sin(\alpha-\beta)+\sin(\alpha+\beta)=2\sin\alpha\cos\beta,$$

or

$$\sin\alpha\cos\beta=\frac{1}{2}(\sin(\alpha-\beta)+\sin(\alpha+\beta)).$$

We will use both these identities in the problem.

$$(\sin 10^\circ)(\sin 30^\circ)(\sin 50^\circ)(\sin 70^\circ)$$

$$\frac{1}{2}(\sin 10^\circ)(\sin 50^\circ)(\sin 70^\circ)$$

$$\frac{1}{4}(\sin 10^\circ)(\cos 20^\circ-\cos 120^\circ)$$

$$\frac{1}{4}(\sin 10^\circ\cos 20^\circ+\frac{1}{2}\sin 10^\circ)$$

$$\frac{1}{4}\left(\frac{1}{2}(\sin(-10^\circ)+\sin 30^\circ)+\frac{1}{2}\sin 10^\circ\right)$$

$$\frac{1}{4}\left(-\frac{1}{2}(\sin(10^\circ))+\frac{1}{4}+\frac{1}{2}\sin 10^\circ\right)$$

$$\frac{1}{4}\left(\frac{1}{4}\right)$$

$$\boxed{\frac{1}{16}}$$