CT ARML Team, 2022
Team Selection Test 1

1. The product of two positive integers is 132300. Compute the maximum possible value of the greatest common divisor of the two integers.
[Answer: 210]

2. The roots of the equation $32x^3 - 48x^2 - 26x + 21 = 0$ are in arithmetic progression. The difference between the largest and the smallest of the three roots is $\frac{a}{b}$, where *a* and *b* are relatively prime positive integers. Find $a + b$.
[Answer: 7]

3. There exists a digit *B* such that, for any digit *A*, the seven-digit number $\underline{1}\,\underline{2}\,\underline{3}\,\underline{A}\,\underline{5}\,\underline{B}\,\underline{7}$ is not divisible by 11. Compute the digit *B*.
[Answer: 4]

4. A video game simulates the motion of a tiny, perfectly bouncy ball. The ball is projected from a vertex of an equilateral triangle of side length 1, and bounces in the interior of the triangle without stopping. Suppose now that the ball bounces off the sides of the triangle exactly 5 times before hitting a vertex of the triangle for the first time. Compute the square of the distance moved by the ball between projection and arrival at the vertex.
[Answer: 13]

5. Suppose that a point is randomly selected from the interior of a right triangle whose legs have length $2\sqrt{3}$ and 4. The probability that the distance of the point from its nearest vertex is less than 2 is $a + \pi\sqrt{b}$, where *a* and *b* are rational numbers. Find $\frac{1}{ab}$.
[Answer: 108]

6. Let $a_1, a_2, a_3, \ldots$ be an arithmetic sequence, let $b_1, b_2, b_3, \ldots$ be a geometric sequence, and let the sequence $c_1, c_2, c_3, \ldots$ be defined by $c_n = a_n + b_n$ for each positive integer *n*. If $c_1 = 1$, $c_2 = 4$, $c_3 = 15$, and $c_4 = 2$, what is the value of c_5?
[Answer: 61]

7. In the complex plane, the complex numbers 0, *z*, $\frac{1}{z}$, and $z + \frac{1}{z}$ form a parallelogram. If the area of the parallelogram is $\frac{35}{37}$, the smallest possible value of $\left|z + \frac{1}{z}\right|^2$ is $\frac{a}{b}$, where *a* and *b* are relatively prime positive integers. Find $a + b$.
[Answer: 87]

8. In square *PQRS* with diagonal of length 1, point *E* is on side $\overline{PQ}$ and point *F* is on side $\overline{QR}$, with $\text{m}\angle QRE = \text{m}\angle QPF = 30°$. Let *G* be the point of intersection of $\overline{RE}$ and $\overline{PF}$. The distance between the incenters of triangles *PGE* and *RGF* is $a - b\sqrt{c}$, where *a*, *b*, *c* are positive integers and *c* is not divisible by the square of any prime number. Find $a + b + c$.
[Answer: 9]

9. Let *R* be the region defined by the inequality $x^2 + y^2 \leq |x| + |y|$, and let the area of region *R* be *A*. Find the integer closest to 1000*A*.
[Answer: 5142]

10. Suppose that $6\tan^{-1} x + 4\tan^{-1}(3x) = \pi$. Then $x^2 = \frac{a - b\sqrt{c}}{d}$, where *a*, *b*, *c*, *d* are integers, $\gcd(a, b, d) = 1$, and *c* is not divisible by the square of any prime number. Find $a + b + c + d$.
[Answer: 59]

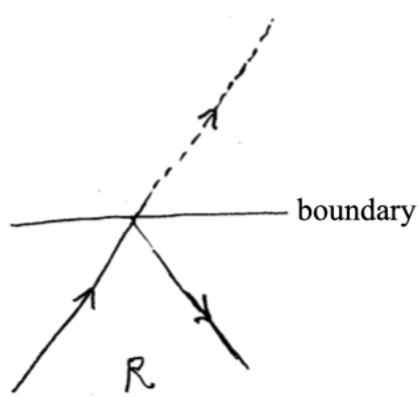
CT ARML Team, 2022
Team Selection Test 1
Solutions

1. First note that $132300 = 2^2 \cdot 3^3 \cdot 5^2 \cdot 7^2$. We need this number to be the product of positive integers m and n. So, $m = 2^{a_1} \cdot 3^{b_1} \cdot 5^{c_1} \cdot 7^{d_1}$ and $n = 2^{a_2} \cdot 3^{b_2} \cdot 5^{c_2} \cdot 7^{d_2}$, where all the exponents are nonnegative integers and $a_1 + a_2 = 2$, $b_1 + b_2 = 3$, and so on. Then, $\gcd(m,n) = 2^{\min\{a_1,a_2\}} \cdot 3^{\min\{b_1,b_2\}} \cdot 5^{\min\{c_1,c_2\}} \cdot 7^{\min\{d_1,d_2\}}$. Looking through the possibilities for $a_1, a_2, b_1, b_2, \dots$, it is easy to see that the maximum possible value of $\min\{a_1,a_2\}$ is 1. Similarly, the maximum values of $\min\{b_1,b_2\}$, $\min\{c_1,c_2\}$, and $\min\{d_1,d_2\}$ are all 1. Thus, the maximum possible value of $\gcd(m,n)$ is $2 \cdot 3 \cdot 5 \cdot 7 = 210$.

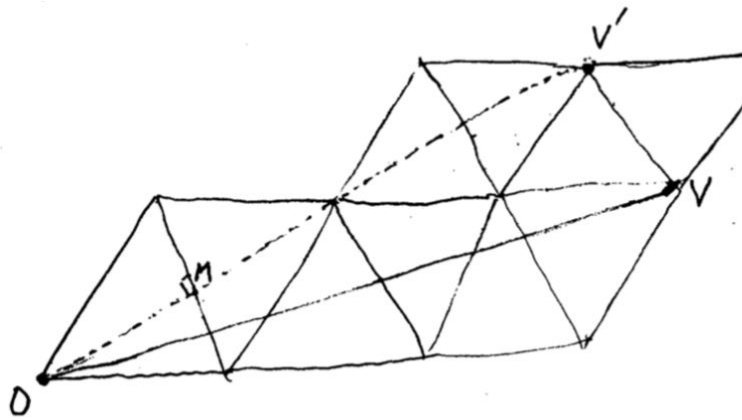
2. Since the roots are in arithmetic progression, they may be written as $a-d, a, a+d$, where $d > 0$. Using Vieta's formula for the sum of the roots we see that $(a-d) + a + (a+d) = \frac{48}{32}$. From this we find that $a = \frac{1}{2}$. Using Vieta's formula for the product of the roots we see that $a(a-d)(a+d) = a(a^2 - d^2) = -\frac{21}{32}$. Using $a = \frac{1}{2}$ and solving for *d* we find that $d = \frac{5}{4}$. Thus, the difference between the largest and smallest roots is $2d = \frac{5}{2}$. So, the answer to the question is 5 + 2 = 7.

3. The given number is divisible by 11 if and only if $7 - B + 5 - A + 3 - 2 + 1$ is divisible by 11; that is, $14 - A - B$ is divisible by 11. This is true if and only if $A + B \equiv 3 \pmod{11}$. We need the digit *B* such that there is no digit *A* for which this is true. Trial and error shows that the only such digit *B* is 4. (If you are unfamiliar with the notation used in this solution, please look up *modular arithmetic*.)

4. The first thing to note is that if a particle is bouncing off a straight boundary of a region *R*, the motion of the particle after the bounce can alternatively be considered in the reflection of the region *R* through the boundary. The motion of the particle before and after the bounce is then a straight line. Please see the diagram below.

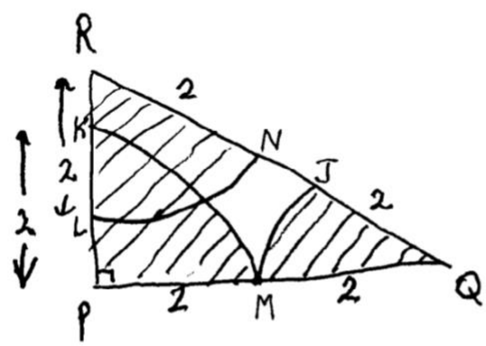


We can therefore consider the motion of the ball in this question as straight-line motion in the plane, with the plane tiled with equilateral triangles. The ball is projected from some vertex *O* of one of the triangles. We will argue that the ball must be projected directly towards the vertex *V* shown in the diagram below, or some other vertex whose distance from *O* is equal to that of *V*.

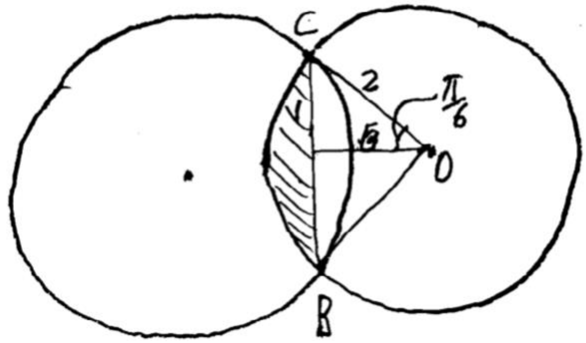


For consider the midpoint *M* of the side of the triangle towards which the ball is projected. If the ball is projected directly towards *M* then the ball only bounces once before hitting a vertex. So, we can assume, without loss of generality, that the ball must be projected strictly to the right of *M*. If the ball is projected directly towards *V,* then it crosses exactly five edges before hitting the vertex *V*, as required. If the ball is projected strictly to the right of *V,* then it is easy to see that it must pass through more than five boundaries before hitting a vertex. If the ball is projected strictly to the right of *M* and strictly to the left of *V* then it is easy to see by sampling the various possible paths that it must pass through fewer, or more, than five edges before hitting a vertex. Therefore, we can assume that the ball is projected towards *V*. Note that $OV'V$ is a right triangle of which $\overline{OV}$ is the hypotenuse. Furthermore, $OV' = 2\sqrt{3}$ and $V'V = 1$. Thus, $OV^2 = 12 + 1 = 13$.

5. The diagram below shows the given triangle, along with three arcs of circles, each of radius 2. We require the shaded area divided by the area of the triangle.



Let the area of the region shared by sectors *RLN* and *PKM* be *A*. The shaded area is the area of sector *RLN* + the area of sector *PKM* + the area of sector *QMJ* – *A*. In order to find *A*, we consider two circles of radius 2 whose centers are $2\sqrt{3}$ apart, as shown in the diagram below.



The shaded area is the area of sector *OBC* – the area of triangle *OBC*

$$= \frac{1}{2} \cdot \frac{\pi}{3} \cdot 2^2 - \frac{1}{2} \cdot 2 \cdot \sqrt{3} = \frac{2\pi}{3} - \sqrt{3}.$$

Therefore, the area of the region common to the interiors of the two circles is $2(\frac{2\pi}{3} - \sqrt{3})$.

Consequently, $A = \frac{1}{2} \cdot 2\left(\frac{2\pi}{3} - \sqrt{3}\right) = \frac{2\pi}{3} - \sqrt{3}.$

Returning to the first diagram, note that, in triangle *PQR*, angles *R* and *Q* are complementary. Thus, the area of sector *RLN* + the area of sector *QMJ* is a quarter of the area of a circle of radius 2, that is, $\frac{1}{4}\pi \cdot 2^2 = \pi$. Further, the area of sector *PKM* = $\frac{1}{4}\pi \cdot 2^2 = \pi$.

Thus, the required probability is

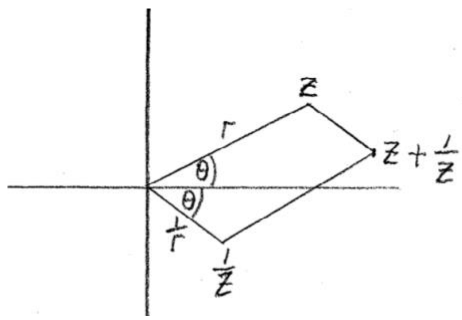
$$\frac{\pi + \pi - \left(\frac{2\pi}{3} - \sqrt{3}\right)}{\frac{1}{2} \cdot 4 \cdot 2\sqrt{3}} = \frac{1}{4} + \pi\sqrt{\frac{1}{27}}\ ,$$

and the answer to the question is $\frac{1}{\left(\frac{1}{4}\right)\left(\frac{1}{27}\right)} = 108.$

6. We can write the arithmetic sequence as $a, a+d, a+2d, \ldots$ and the geometric sequence as $b, br, br^2, \ldots$. Then, from the information given we can write
$a + b = 1$ (1)
$a + d + br = 4$ (2)
$a + 2d + br^2 = 15$ (3)
$a + 3d + br^3 = 2$ (4)
Subtracting equation (1) from equation (2) we get $a + b(r-1) = 3.$ (5)
Subtracting equation (2) from equation (3) we get $d + br(r-1) = 11.$ (6)
Subtracting equation (3) from equation (4) we get $d + br^2(r-1) = -13.$ (7)
Subtracting equation (5) from equation (6) we get $b(r-1)^2 = 8.$ (8)
Subtracting equation (6) from equation (7) we get $br(r-1)^2 = -24.$ (9)
Dividing equation (9) by equation (8) we get $r = -3.$

Substituting this into equation (8) we get $b = \frac{1}{2}$.
Substituting this into equation (1) we get $a = \frac{1}{2}$.
Substituting these three values into equation (2) we get $d = 5$.
Thus, $c_5 = a + 4d + br^4 = 61$.

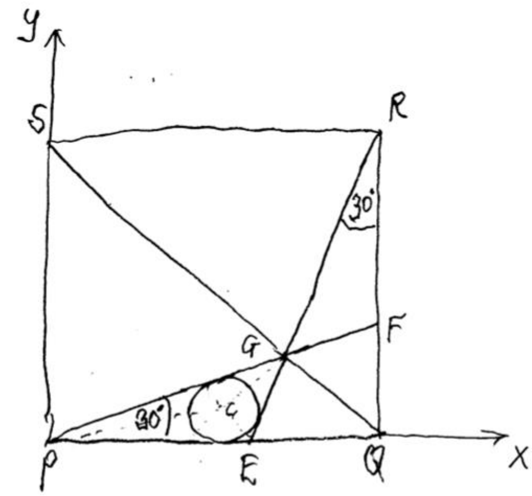
7. Let the modulus and argument of z be r and θ, respectively. Then, by de Moivre's theorem, the modulus and argument of $\frac{1}{z}$ are $\frac{1}{r}$ and $-\theta$, as shown in the diagram below.



As stated in the question, the complex numbers 0, z, $\frac{1}{z}$, and $z + \frac{1}{z}$ form a parallelogram. Using the "$\frac{1}{2}ab\sin C$" formula for the area of a triangle, the area of this parallelogram is $r \cdot \frac{1}{r}\sin 2\theta$ $= \sin 2\theta$. Thus, $\sin 2\theta = \frac{35}{37}$. (We can assume that $0 \le \theta \le \frac{\pi}{2}$, as this supplies all possibilities for the angle between z and $\frac{1}{z}$.) It follows that $\cos^2 2\theta = 1 - \sin^2 2\theta = 1 - \left(\frac{35}{37}\right)^2 = \left(\frac{12}{37}\right)^2$. So, $\cos 2\theta = \pm\frac{12}{37}$.

Note that the angles in the parallelogram not equal to 2θ are $\pi - 2\theta$. So, using the law of cosines, $\left|z + \frac{1}{z}\right|^2 = r^2 + \left(\frac{1}{r}\right)^2 - 2r \cdot \frac{1}{r}\cos(\pi - 2\theta) = r^2 + \frac{1}{r^2} + 2\cos 2\theta$. Using the fact that $r^2 + \frac{1}{r^2} = \left(r - \frac{1}{r}\right)^2 + 2$, we see that the minimum value of $r^2 + \frac{1}{r^2}$ is 2. The minimum value of $\cos 2\theta$ is $-\frac{12}{37}$. Thus, the minimum value of $\left|z + \frac{1}{z}\right|^2$ is $2 - 2\left(\frac{12}{37}\right) = \frac{50}{37}$, and the answer to the question is $50 + 37 = 87$.

8. Let the incenter of triangle PEG be C and the incenter of triangle RGF be C'. Establish x- and y-axes as shown in the diagram below.



Using the Pythagorean theorem and trigonometry is it straightforward to show that $RQ = \frac{1}{\sqrt{2}}$ and $EQ = \frac{1}{\sqrt{6}}$. It follows that the coordinates of E are $\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}}, 0\right)$.

The equation of line PF is $y = (\tan 30°)x = \frac{1}{\sqrt{3}}x$ and the equation of line SQ is $x + y = \frac{1}{\sqrt{2}}$. Solving these equations as a system, we find that the coordinates of G are $\left(\frac{\sqrt{3}}{\sqrt{2}(\sqrt{3}+1)}, \frac{1}{\sqrt{2}(\sqrt{3}+1)}\right)$.

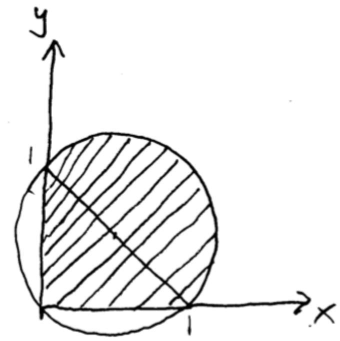
Point C lies at the intersection of the angle bisectors of triangle PEG. Thus, the equation of line PC is $y = x \tan 15° = x \tan(60° - 45°) = x \cdot \frac{\sqrt{3}-1}{1+\sqrt{3}}$. Note that line EC is inclined at 60° to line EP. Thus, the equation of line EC is $y - 0 = -(\tan 60°)\left(x - \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}}\right)\right)$; that is, $y = -\sqrt{3}\left(x - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{6}}\right)$. Solving these equations as a system we learn that the x-coordinate of C is $\frac{\sqrt{3}-1}{2\sqrt{2}}$.

Note, now, that $CC' = 2 \cdot CG = 2 \cdot \sqrt{2} \cdot$ (the difference between the x-coordinates of G and C).

Thus, $CC' = 2\sqrt{2}\left(\frac{\sqrt{3}}{\sqrt{2}(\sqrt{3}+1)} - \frac{\sqrt{3}-1}{2\sqrt{2}}\right) = 4 - 2\sqrt{3}$, and the answer to the question is 4 + 2 + 3 = 9.

9. The given inequality can be written as $|x|^2 + |y|^2 \leq |x| + |y|$. Therefore, the region R is symmetrical about the y-axis and about the x-axis. It follows that the area of R is four times the area of the portion of R that lies in the first quadrant, which we will call R_1.

For points in the first quadrant the inequality may be written as $x^2 + y^2 \leq x + y$. This can be written as $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 \leq \left(\frac{1}{\sqrt{2}}\right)^2$. It follows that R_1 is the shaded portion of the interior of the circle shown in the diagram below. The center of the circle is $\left(\frac{1}{2}, \frac{1}{2}\right)$ and its radius is $\frac{1}{\sqrt{2}}$.



Since R_1 consists of a right triangle and a semicircle we have that

$$\text{Area}(R_1) = \frac{1}{2} \cdot 1 \cdot 1 + \frac{1}{2}\pi\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{4}\pi.$$

Hence, $A = 4 \cdot \text{Area}(R_1) = 2 + \pi \approx 5.142$, and the integer closest to $1000A$ is 5142.

10. The given equation tells us that $3\tan^{-1} x + 2\tan^{-1} 3x = \frac{\pi}{2}$. Hence,
$3\tan^{-1} x = \frac{\pi}{2} - 2\tan^{-1} 3x.$ (1)

We will now use the facts that
$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ (2)
and $\tan 3\theta = \tan(2\theta + \theta) = \frac{(\tan 2\theta + \tan\theta)}{1 - \tan 2\theta \tan\theta} = \frac{\frac{2\tan\theta}{1-\tan^2\theta} + \tan\theta}{1 - \frac{2\tan^2\theta}{1-\tan^2\theta}} = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$, (3)

along with the fact that $\tan\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\tan\theta}$. (4)

Taking the tan of both sides of (1) and using (4) we have $\tan(3\tan^{-1} x) = \frac{1}{\tan(2\,tan^{-1} 3x)}$. Then, using (2) and (3), $\frac{3x - x^3}{1 - 3x^2} = \frac{1 - 9x^2}{6x}$. This equation reduces to $33x^4 - 30x^2 + 1 = 0$, telling us that $x^2 = \frac{15 \pm \sqrt{192}}{33} = \frac{15 \pm 8\sqrt{3}}{33}$.

We will now show that $x^2 = \frac{15+\sqrt{192}}{33}$ isn't possible. For, then, x^2 is a bit less than 1, so x is a bit less than 1, so $\tan^{-1} x$ is a bit less than 45° (using degrees for convenience), and $\tan^{-1} 3x$ is an angle between 45° and 90°. Then it's impossible that $3\tan^{-1} x + 2\tan^{-1} 3x = 90°$, as is required by the question.

Hence, $x^2 = \frac{15-8\sqrt{3}}{33}$, and the answer to the question is 15 + 8 + 3 + 33 = 59.

1 Team Problems

Problem 1. Compute the greatest integer x such that $\left\lfloor \sqrt{\left\lfloor \sqrt{\lfloor \sqrt{x} \rfloor} \right\rfloor} \right\rfloor = 2$.

Problem 2. Compute the number of ordered pairs of integers (x, y) such that $\frac{1}{x} + \frac{540}{xy} = 2$.

Problem 3. A positive integer has the *Kelly Property* if it contains a zero in its base-17 representation. Compute the number of positive integers less than 1000 (base 10) that have the Kelly Property.

Problem 4. Compute the smallest positive integer n such that $n + i$, $(n + i)^2$, and $(n + i)^3$ are the vertices of a triangle in the complex plane whose area is greater than 2015.

Problem 5. A loop is made by connecting rods of lengths $1, 2, \ldots, 90$ in that order. (The rod of length 90 is connected to the rods of lengths 89 and 1.) The loop is laid in the shape of an equilateral triangle of perimeter 4095. Rods cannot be bent or broken. Compute the sum of the lengths of the shortest rods on each side of the triangle. (For example, the loop with rods 1, 2, ..., 9 can be arranged into an equilateral triangle because $4 + 5 + 6 = 7 + 8 = 9 + 1 + 2 + 3$.)

Problem 6. Four spheres S_1, S_2, S_3, and S_4 are mutually externally tangent and are tangent to a plane, on the same side of the plane. Let S_1 and S_2 have radius r and let S_3 and S_4 have radius s. Given that $r > s$, compute r/s.

Problem 7. In $\triangle ABC$, point D is on $\overline{AB}$ and point E is on $\overline{AC}$. The measures of the nine angles in triangles ADE, BCD, and CDE can be arranged to form an arithmetic sequence. Compute the greatest possible degree measure for $\angle A$.

Problem 8. Let $a_1 = a_2 = a_3 = 1$. For $n > 3$, let a_n be the number of real numbers x such that

$$x^4 - 2a_{n-1}x^2 + a_{n-2}a_{n-3} = 0.$$

Compute the sum $a_1 + a_2 + a_3 + \cdots + a_{1000}$.

Problem 9. For any real number k, let region R_k consist of all points (x, y) such that $x \geq 0$, $y \geq 0$, and $\lfloor x + y \rfloor + \{x\} \leq k$, where $\{u\}$ denotes the fractional part of u. Compute the value of k for which the area of R_k is equal to 100.

Problem 10. Let $ABCD$ be a parallelogram with $m\angle A > 90°$. Point E lies on $\overrightarrow{DA}$ such that $\overline{BE} \perp \overline{AD}$. The circumcircles of $\triangle ABC$ and $\triangle CDE$ intersect at points F and C. Given that $AD = 35$, $DC = 48$, and $CF = 50$, compute AC.

2 Answers to Team Problems

Answer 1. 6560

Answer 2. 15

Answer 3. 106

Answer 4. 9

Answer 5. 72

Answer 6. $2 + \sqrt{3}$

Answer 7. 84

Answer 8. 2329

Answer 9. $29 - \sqrt{221}$

Answer 10. $5\sqrt{23}$

3 Solutions to Team Problems

Problem 1. Compute the greatest integer x such that $\left\lfloor\sqrt{\left\lfloor\sqrt{\lfloor\sqrt{x}\rfloor}\right\rfloor}\right\rfloor = 2$.

Solution 1. Ignore the floor functions and replace 2 with 3. Then x would be $((3^2)^2)^2 = 6561$. If $x < 6561$, then $\sqrt{x} < 81$, so $\lfloor\sqrt{x}\rfloor \leq 80$, and $\sqrt{\lfloor\sqrt{x}\rfloor} < 9$, so $\left\lfloor\sqrt{\lfloor\sqrt{x}\rfloor}\right\rfloor \leq 8$, and $\left\lfloor\sqrt{\left\lfloor\sqrt{\lfloor\sqrt{x}\rfloor}\right\rfloor}\right\rfloor \leq 2$. Thus the greatest possible integral value of x is $6561 - 1 = \mathbf{6560}$.

Problem 2. Compute the number of ordered pairs of integers (x, y) such that $\frac{1}{x} + \frac{540}{xy} = 2$.

Solution 2. Multiply through by xy and rearrange terms to obtain $540 = 2xy - y = (2x - 1) \cdot y$. Because $2x - 1$ is odd, the question reduces to counting the number of odd divisors (positive and negative) of 540. Because $540 = 2^2 \cdot 3^3 \cdot 5$, there are $(3+1)(1+1) = 8$ positive odd divisors, hence 540 has 16 odd integer divisors and 16 potential integral values for x. However, the divisor -1 corresponds to $x = 0$, which is extraneous to the original equation. Hence there are **15** solutions, given in the tables below.

$\boldsymbol{2x-1}$	-135	-45	-27	-15	-9	-5	-3
$\boldsymbol{x}$	-67	-22	-13	-7	-4	-2	-1
$\boldsymbol{y}$	-4	-12	-20	-36	-60	-108	-180

$\boldsymbol{2x-1}$	1	3	5	9	15	27	45	135
$\boldsymbol{x}$	1	2	3	5	8	14	23	68
$\boldsymbol{y}$	540	180	108	60	36	20	12	4

Problem 3. A positive integer has the *Kelly Property* if it contains a zero in its base-17 representation. Compute the number of positive integers less than 1000 (base 10) that have the Kelly Property.

Solution 3. Compute the base-17 representation of 1000:

$$1000 = 3 \cdot 17^2 + 7 \cdot 17^1 + 14 \cdot 17^0.$$

Let y_p denote the number of integers less than 1000 with the Kelly Property whose 17^2-digit is p. If $p = 1$ or $p = 2$, then so long as one of the two rightmost digits is zero, the number will have the Kelly Property. There are $17^2 - 16^2$ such numbers because 16^2 of the 17^2 options have non-zero values in both of the two rightmost digits. Thus $y_1 = y_2 = 17^2 - 16^2 = 33$. For $p = 0$, the only possibility for a number with the Kelly Property is for its 1s digit to be zero. There are 16 such numbers, because the 17s digit can be any non-zero value. Thus $y_0 = 16$. Finally consider the case $p = 3$. If the 17s digit is zero, then any value for the 1s digit will yield a number with the Kelly Property, contributing another 17 numbers with the Kelly Property. If the 17s digit is a number 1 through 7 inclusive, then to have the Kelly Property, the number must have a zero in its 1s digit, contributing another 7 numbers with the Kelly Property. Thus $y_3 = 17 + 7 = 24$.

The number of positive integers less than 1000 which have the Kelly Property is therefore

$$y_0 + y_1 + y_2 + y_3 = 16 + 33 + 33 + 24 = \mathbf{106}.$$

Problem 4. Compute the smallest positive integer n such that $n+i$, $(n+i)^2$, and $(n+i)^3$ are the vertices of a triangle in the complex plane whose area is greater than 2015.

Solution 4. The complex number $(n+i)^2$ can be broken into real and imaginary parts as $n^2+2ni+i^2 = (n^2-1)+2ni$. The complex number $(n+i)^3$ can broken into real and imaginary parts as $n^3+3n^2i+3ni^2+i^3 = (n^2-3n)+(3n^2-1)i$. Therefore the triangle has the same area in the complex plane as the triangle in the Cartesian plane with coordinates $(n,1)$, $(n^2-1, 2n)$, and $(n^3-3n, 3n^2-1)$. The Shoelace Theorem gives that the area of this triangle is the absolute value of

$$\begin{aligned} &\frac{1}{2}\begin{vmatrix} n & 1 \\ n^2-1 & 2n \end{vmatrix} + \frac{1}{2}\begin{vmatrix} n^2-1 & 2n \\ n^3-3n & 3n^2-1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} n^3-3n & 3n^2-1 \\ n & 1 \end{vmatrix} \\ =\ &\frac{1}{2}(2n^2+3n^4-n^2-3n^2+1+n^3-3n-n^2+1-2n^4+6n^2-3n^3+n) \\ =\ &\frac{1}{2}(n^4-2n^3+3n^2-2n+2). \end{aligned}$$

Therefore the problem reduces to finding the smallest positive n such that $f(n) = \left|n^4-2n^3+3n^2-2n+2\right| > 4030$. Notice that the dominant term is n^4, and so $n=8$ gives $n^4=4096$, which is a good approximation to 4030. However, substituting $n=8$ gives $f(n) = |4096-2\cdot 512+3\cdot 64-2\cdot 8+2| = 3250$, which is too small. Trying $n=9$ gives $f(n) = |6561-2\cdot 729+3\cdot 81-2\cdot 9+2| = 5330 > 4030$, so the answer is $n = \mathbf{9}$.

Alternate Solution: Let $z=(n+i)$, and let K_1 be the area of the triangle whose vertices are 0, z, and z^2, let K_2 be the area of the triangle whose vertices are 0, z^2, and z^3, and let K_3 be the area of the triangle whose vertices are 0, z, and z^3. Then if K is the area of the triangle with vertices z, z^2, and z^3, $K = K_1+K_2-K_3$.

First compute K_1. Let θ be the angle that $n+i$ makes with the x-axis. By DeMoivre's Theorem, the angle between the ray from 0 through z and the ray from 0 through z^2 is also θ, so using the triangle area formula $\frac{1}{2}ab\sin C$ yields

$$K_1 = \frac{1}{2}\cdot|n+i|\cdot\left|(n+i)^2\right|\cdot\sin\theta.$$

Because absolute values are multiplicative, the expression on the right-hand side simplifies to

$$K_1 = \frac{1}{2}\cdot|n+i|^3\cdot\sin\theta,$$

and because $|n+i| = \sqrt{n^2+1}$,

$$K_1 = \frac{1}{2}\cdot(n^2+1)^{3/2}\sin\theta.$$

Now $\sin\theta = \frac{1}{\sqrt{n^2+1}}$, so

$$K_1 = \frac{n^2+1}{2}.$$

Similarly, $K_2 = \frac{(n^2+1)^2}{2}$.

Computing K_3 is not much more involved. The angle between z and z^3 is 2θ. Using the trigonometric identity $\sin 2x = 2\sin x\cos x$ and the observation that $\cos\theta = \frac{n}{\sqrt{n^2+1}}$ yields

$$\begin{aligned} \sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\left(\frac{1}{\sqrt{n^2+1}}\right)\left(\frac{n}{\sqrt{n^2+1}}\right) \\ &= \frac{2n}{n^2+1}. \end{aligned}$$

Thus

$$K_3 = \frac{1}{2}\cdot(n^2+1)^2\cdot\frac{2n}{n^2+1} = n(n^2+1).$$

Hence

$$\begin{aligned}
K &= K_1 + K_2 - K_3 \\
&= \frac{n^2+1}{2} + \frac{(n^2+1)^2}{2} - n(n^2+1) \\
&= \left(n^2+1\right)\left(\frac{1}{2} + \frac{n^2+1}{2} - n\right) \\
&= \left(n^2+1\right)\left(\frac{n^2-2n+2}{2}\right) \\
&= \frac{1}{2}\left(n^2+1\right)\left((n-1)^2+1\right).
\end{aligned}$$

To compute the smallest positive integral value of n such that $K > 2015$, first multiply by 2 to obtain $\left(n^2+1\right)\left((n-1)^2+1\right) > 4030$. The two factors are approximately n^2 and $(n-1)^2$, so look for n such that $n^4 > 4030$. Because $8^4 = 2^{12} = 4096$, try $n = 8$ to obtain a product of $(64+1)(49+1) = 3250$ which is too small; $n = 9$ yields a product of $(81+1)(64+1) = 5330$. Thus $n = \mathbf{9}$.

Problem 5. A loop is made by connecting rods of lengths $1, 2, \ldots, 90$ in that order. (The rod of length 90 is connected to the rods of lengths 89 and 1.) The loop is laid in the shape of an equilateral triangle of perimeter 4095. Rods cannot be bent or broken. Compute the sum of the lengths of the shortest rods on each side of the triangle. (For example, the loop with rods 1, 2, $\ldots$, 9 can be arranged into an equilateral triangle because $4+5+6 = 7+8 = 9+1+2+3$.)

Solution 5. The side length of the triangle is $\frac{1}{3} \cdot \frac{91 \cdot 90}{2} = 91 \cdot 15 = 3 \cdot 5 \cdot 7 \cdot 13 = 1365$. Call this value S. Start walking around the triangle at the length-1 rod, going in the direction away from the length-90 rod. Let the last rods of each of the three sides have lengths a, b, and c, with $a < b < c$. (In the 9-rod example, $(a, b, c) = (3, 6, 8)$). Thus the three sides are composed of rods $a+1, \ldots, b$; $b+1, \ldots c$; and $c+1, \ldots, 90, 1, \ldots, a$.

Focus on the side that ends in the length-c rod. It has $c - b$ rods in total, and $S = (c-b)(\frac{b+c+1}{2})$. If $c - b$ is odd, then it divides S. If $c - b$ is even, then it divides $2S$. If $c - b \le 15$, then $\frac{S}{c-b}$ — the average length of the rods on that side — will be at least 91, which is too big.

If $c - b = 21$, then $\frac{S}{c-b} = 65$, so this side must have rods $55, \ldots, 75$. That would imply $b = 54$. Does there exist an a such that $S = (54-a)(\frac{a+54+1}{2})$? As it turns out, $a = 15$ is a solution. Thus $(a, b, c) = (15, 54, 75)$ produces an equilateral triangle.

If $c - b = 26$, then $\frac{S}{c-b} = 52.5$, so this side must have rods $40, \ldots, 65$. But note that this means the side *after* this must contain rods 66 through 90, as well as the length-1 rod and possibly more after that. The total length of those rods is at least $78 \cdot 25 + 1 > S$, which is too large.

Thus for all values of $c - b$ that divide $2S$ *except* 21, there will not exist an equilateral triangle satisfying the conditions of the problem. Thus the only possible value for the sum of the lengths of the shortest rods of the three sides is $1 + 16 + 55 = \mathbf{72}$.

Remark: This problem admits several interesting generalizations, three of which are discussed below.

No squares. If $(A, B, C) = (2a+1, 2b+1, 2c+1)$, then it must be the case that $8S = B^2 - A^2 = C^2 - B^2$. That is, A^2, B^2, C^2 must form an arithmetic progression. In general, to create a regular k-gon with the n-loop (rods of lengths $1, 2, \ldots, n$), then it is necessary to be able to create arithmetic progressions of k odd perfect squares. It so happens that there are no such progressions for $k \ge 4$, a result shown by Euler. Therefore it is not possible make the loop into a square (or larger k-gon) for *any* integer n.

Equilateral triangles are hard to find. The values of n less than 1000 for which the n-loop can be made into an equilateral triangle are 9, 90, 125, 153, 189, 440, 819, and 989. In the $n = 125$ case, there are two different ways to create an equilateral triangle! Can you determine whether any other value n has this property?

There are infinitely many such equilateral triangles, though. One can use Diophantine equations to find an infinite family of values of n that allow for equilateral triangles.

Problem 6. Four spheres S_1, S_2, S_3, and S_4 are mutually externally tangent and are tangent to a plane, on the same side of the plane. Let S_1 and S_2 have radius r and let S_3 and S_4 have radius s. Given that $r > s$, compute r/s.

Solution 6. Let the centers of S_1 and S_2 be at $(r, r, 0)$ and $(-r, r, 0)$, so that they are tangent to the plane $y = 0$ at points $(r, 0, 0)$ and $(-r, 0, 0)$. Then the centers of S_3 and S_4 lie on the plane $x = 0$, at $(0, s, -s)$ and $(0, s, s)$. Because all of the spheres are mutually tangent, the distance between the center of S_1 and the center of S_4 is $r + s$, so the 3-dimensional Pythagorean Theorem gives

$$(r + s)^2 = r^2 + (r - s)^2 + s^2$$

which simplifies to

$$r^2 - 4rs + s^2 = 0.$$

Solving this as a quadratic in r gives

$$r = (2 \pm \sqrt{3})s.$$

Because $r > s$, it follows that $\frac{r}{s} = \mathbf{2 + \sqrt{3}}$.

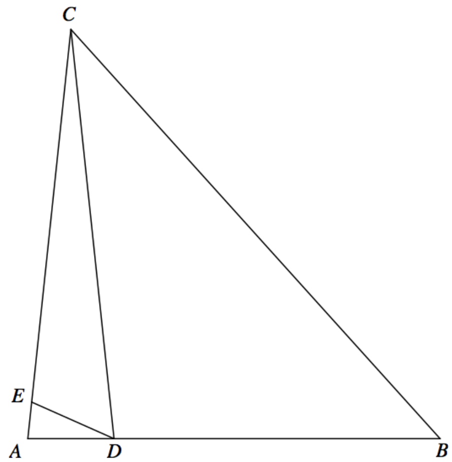
Alternate Solution: Let ω be the given plane, and let the centers of the four spheres be P_1, P_2, P_3, and P_4 respectively. Because S_1 and S_2 have equal radii and S_3 and S_4 have equal radii, note that $\overline{P_1P_2}$ and $\overline{P_3P_4}$ are parallel to ω. Note also that $P_1P_2 = 2r$, $P_3P_4 = 2s$, and $P_1P_3 = P_1P_4 = P_2P_3 = P_2P_4 = r + s$. Let R be the midpoint of $\overline{P_3P_4}$, and let Q denote the point in the plane parallel to ω containing P_3 and P_4, such that $\overleftrightarrow{QP_1} \perp \omega$. Then $\triangle QRS_3$ is a right triangle with legs of length r and s, and hypotenuse $QS_3 = \sqrt{r^2 + s^2}$. But $\triangle QP_1P_3$ is also a right triangle, with hypotenuse $\overline{P_1P_3}$. Because $QP_1 = r - s$, the Pythagorean Theorem yields $(r - s)^2 + r^2 + s^2 = (r + s)^2$. Hence $r^2 + s^2 = 4rs$. Divide by s^2 to obtain the equation $\frac{r^2}{s^2} + 1 = 4 \cdot \frac{r}{s}$; substitute $u = \frac{r}{s}$ to obtain the quadratic equation $u^2 - 4u + 1 = 0$. Thus $u = 2 \pm \sqrt{3}$. It is given that $r > s$, hence $\frac{r}{s} = \mathbf{2 + \sqrt{3}}$.

Problem 7. In $\triangle ABC$, point D is on $\overline{AB}$ and point E is on $\overline{AC}$. The measures of the nine angles in triangles ADE, BCD, and CDE can be arranged to form an arithmetic sequence. Compute the greatest possible degree measure for $\angle A$.

Solution 7. The average value of the angles is $60°$, so the nine terms must be $60° \pm nd$, where d is the common difference and $0 \leq n \leq 4$. Thus $d < 15°$. Looking at the supplementary angles at E, one angle is $60° + kd$, and the other is $60° + \ell d$. Let $m = k + \ell$. Then $d = \frac{60°}{m}$. Combining these facts yields only three possible values for m, namely, $m \in \{5, 6, 7\}$.

If $d = 10°$, then the supplementary angles must be $80°$ and $100°$, in some order, and so $\mathrm{m}\angle A \leq 90°$. But then $\mathrm{m}\angle ADE = 10°$, which is impossible. One can successfully fill the diagram with $\mathrm{m}\angle A = 70°$, which is the maximum for $d = 10°$.

If $d = 12°$, then the supplementary angles can be $\{72°, 108°\}$ or $\{84°, 96°\}$. Note that $\mathrm{m}\angle A$ cannot be $108°$, so try $96°$. Then $\mathrm{m}\angle AED = 72°$ and $\mathrm{m}\angle ADE = 12°$. But $\mathrm{m}\angle CDE + \mathrm{m}\angle BDC = 168°$, and the two largest unused angles are $84°$ and $60°$, so this case is impossible. One can successfully fill the diagram with $\mathrm{m}\angle A = 84°$, as shown below. That is the maximum for this case.



$\triangle BCD$ has angles with measures $(48°, 36°, 96°)$.
$\triangle ADE$ has angles with measures $(84°, 24°, 72°)$.
$\triangle CDE$ has angles with measures $(12°, 60°, 108°)$.

If $d = \frac{60}{7}^{\circ}$, then the largest angles, $\frac{660}{7}^{\circ}$ and $\frac{600}{7}^{\circ}$, must be located at E. The next-largest angle, $\frac{540}{7}^{\circ}$, is less than $84°$. Hence the largest possible angle is $\mathbf{84}°$.

Problem 8. Let $a_1 = a_2 = a_3 = 1$. For $n > 3$, let a_n be the number of real numbers x such that

$$x^4 - 2a_{n-1}x^2 + a_{n-2}a_{n-3} = 0.$$

Compute the sum $a_1 + a_2 + a_3 + \cdots + a_{1000}$.

Solution 8. Consider the quartic equation $x^4 - 2px^2 + q = 0$, where p and q are nonnegative. This equation can be rewritten as $(x^2 - p)^2 = p^2 - q$. Split into cases to determine the number of distinct real roots:

- If $p^2 - q < 0$, there are **0** real roots.
- If $p^2 - q = 0$ **and** $p > 0$, there are **2** real roots, at $x = \pm\sqrt{p}$.
- If $p^2 - q = 0$ **and** $p = 0$, then there is **1** real root, at $x = 0$. (Note that this is just the case $x^4 = 0$.)
- If $p^2 - q > 0$ **and** $q > 0$, then there are **4** real roots, at $x = \pm\sqrt{p \pm \sqrt{p^2 - q}}$.
- If $p^2 - q > 0$ **and** $q = 0$, then there are **3** real roots, at $x = 0$ and $x = \pm\sqrt{2p}$.

To determine a_n, iteratively apply the above rules, with $p = a_{n-1}$ and $q = a_{n-2}a_{n-3}$. Because each term a_n depends only on the three previous values a_{n-1}, a_{n-2}, and a_{n-3}, it suffices to find a group of three consecutive terms that occurs twice.

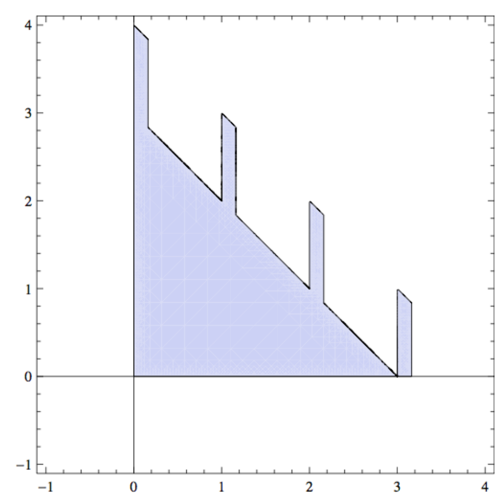
n	$p = a_{n-1}$	$q = a_{n-2}a_{n-3}$	$p^2 - q$	a_n
4	1	1	0	2
5	2	1	3	4
6	4	2	12	4
7	4	8	8	4
8	4	16	0	2
9	2	16	-12	0
10	0	8	-8	0
11	0	0	0	1
12	1	0	1	3
13	3	0	9	3
14	3	3	6	4
15	4	9	7	4
16	4	12	4	4

The 3-term subsequence $4, 4, 4$ occurs starting at a_5 and again at a_{14}. Thus the sequence has the following 9-term period: $4, 4, 4, 2, 0, 0, 1, 3, 3$. The sum of these terms is 21, so the sum $a_1 + a_2 + \cdots + a_{1000}$ is

$$\begin{aligned}\sum_{n=1}^{1000} a_n &= \left(\sum_{n=1}^{4} a_n\right) + \left(\sum_{n=5}^{994} a_n\right) + \left(\sum_{n=995}^{1000} a_n\right)\\ &= (1+1+1+2) + 110 \cdot 21 + (4+4+4+2+0+0)\\ &= \mathbf{2329}.\end{aligned}$$

Problem 9. For any real number k, let region R_k consist of all points (x, y) such that $x \geq 0$, $y \geq 0$, and $\lfloor x + y \rfloor + \{x\} \leq k$, where $\{u\}$ denotes the fractional part of u. Compute the value of k for which the area of R_k is equal to 100.

Solution 9. Let $N = \lfloor k \rfloor$ and let $r = \{k\}$. The region R_k is the triangle $x + y < N$, augmented by vertical "slices" of the region $N \leq x + y < N + 1$. There are N slices that are parallelograms of height 1 and width r, as well as one slice that is a trapezoid of area $r - \frac{r^2}{2}$. The region for $k = \sqrt{10}$ is shown below.



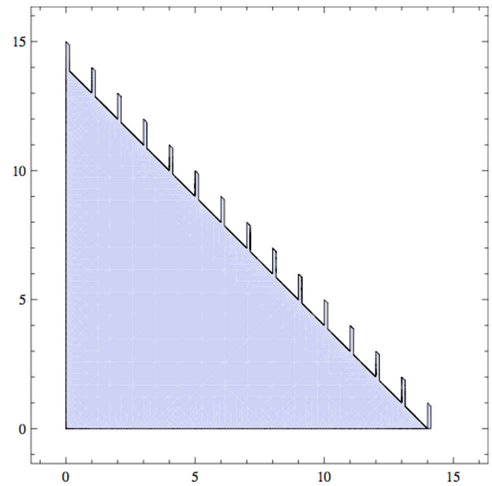
In all, the area is $\frac{N^2 - r^2}{2} + (N + 1)r$. The goal is to compute k so that

$$\frac{N^2 - r^2}{2} + (N + 1)r = 100.$$

Replace r with zero and solve for N to obtain an estimated value for N. This substitution yields $\frac{N^2}{2} = 100$, and so N is approximately $\sqrt{200}$ which is slightly larger than 14. Trying $N = 14$, the equation becomes

$$\frac{14^2 - r^2}{2} + 15r = 100.$$

Simplifying this equation yields $r^2 - 30r + 4 = 0$, and so $r = 15 - \sqrt{221}$. Because $14^2 < 221 < 15^2$, this value of r satisfies $0 < r < 1$. Hence, $k = N + r = \mathbf{29 - \sqrt{221}}$.
The region corresponding to this value of k is shown below.



Problem 10. Let $ABCD$ be a parallelogram with $m\angle A > 90°$. Point E lies on $\overrightarrow{DA}$ such that $\overline{BE} \perp \overline{AD}$. The circumcircles of $\triangle ABC$ and $\triangle CDE$ intersect at points F and C. Given that $AD = 35$, $DC = 48$, and $CF = 50$, compute AC.

Solution 10. Let N lie on $\overline{BC}$ such that $BEAN$ is a rectangle with diagonal $\overline{BA}$. Then $EN = AB = 48$. Trapezoid $ENCD$ is isosceles, and hence a cyclic quadrilateral, so N lies on the circumcircle of $\triangle CDE$. Let G be the intersection of $\overline{BA}$ and $\overline{EN}$. Then $BG = GA = EG = GN = 24$.

Recall that the power of a point P with respect to a circle O is the product of the distance $PX \cdot PY$, where P, X, and Y are collinear, and X and Y are on O; The power of a point P is invariant for any such chord $\overline{XY}$. Because $\overline{BA}$ is a chord of the circumcircle of $\triangle ABC$ and $\overline{EN}$ is a chord of the circumcircle of $\triangle CDE$, the power of point G with respect to both circumcircles is $24 \cdot 24$.

Now recall that given two intersecting circles, the locus of points with the same power in each circle is the line connecting the two points of intersection of the circles, so G lies on $\overline{CF}$.

Applying the power of a point theorem again to chord CF gives $GC \cdot CF = 24 \cdot 24 = 496$, and $GC + CF = 50$, so CG and GF are 18 and 32, in some order.

Apply Stewart's Theorem to triangle ABC to get

$$AC^2 \cdot BG + BC^2 \cdot GA = CG^2 \cdot AB + AG \cdot GB \cdot AB,$$

or

$$AC^2 \cdot 24 + 35^2 \cdot 24 = CG^2 \cdot 48 + 24 \cdot 24 \cdot 48$$

which gives

$$AC = \sqrt{2CG^2 - 73}.$$

The two values for CG then yield $AC = 5\sqrt{79}$ or $AC = \mathbf{5\sqrt{23}}$.