CT ARML Team
Team Selection Test 2, 2022

I-1. Let x be the smallest positive integer such that $1584 \cdot x$ is a perfect cube, and let y be the smallest positive integer such that xy is a multiple of 1584. Compute y. [12]

I-2. In triangle ABC, C is a right angle and M is on $\overline{AC}$. A circle with radius r is centered at M, is tangent to $\overline{AB}$, and is tangent to $\overline{BC}$ at C. If $AC = 5$ and $BC = 12$, then $r = \frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$. [17]

I-3. Some people at a meeting are men, and the rest are women. Selecting from the people at the meeting, there are exactly 25 ways to pick a group of three people that includes at least one person of each gender. Compute the number of people at the meeting. [7]

I-4. Regular hexagon $ABCDEF$ and regular hexagon $GHIJKL$ both have side length 24. The hexagons overlap, so that G is on $\overline{AB}$, B is on $\overline{GH}$, K is on $\overline{DE}$, and D is on $\overline{JK}$. If $[GBCDKL] = \frac{1}{2}[ABCDEF]$, compute LF. [18]

I-5. Let X be the number of digits in the decimal expansion of $100^{1000^{10{,}000}}$, and let Y be the number of digits in the decimal expansion of $1000^{10{,}000^{100{,}000}}$. Compute $\lfloor \log_X Y \rfloor$. (Note: $\lfloor x \rfloor$ is the greatest integer less than or equal to x.) [13]

I-6 Define the sequence of positive integers $\{a_n\}$ as follows:

$$\begin{cases} a_1 = 1; \\ \text{for } n \geq 2,\ a_n \text{ is the smallest possible positive value of } n - a_k^2, \text{ for } 1 \leq k < n. \end{cases}$$

For example, $a_2 = 2 - 1^2 = 1$, and $a_3 = 3 - 1^2 = 2$. Compute $a_1 + a_2 + \cdots + a_{50}$. [253]

I-7. Let A and B be digits from the set $\{0, 1, 2, \ldots, 9\}$. Let r be the two-digit integer $\underline{A}\,\underline{B}$ and let s be the two-digit integer $\underline{B}\,\underline{A}$, so that r and s are members of the set $\{00, 01, \ldots, 99\}$. Compute the number of ordered pairs (A, B) such that $|r - s| = k^2$ for some integer k. [42]

I-8. The *grid distance* between points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ is defined as $d(A, B) = |x_A - x_B| + |y_A - y_B|$. Given some $s > 0$ and points $A = (x_A, y_A)$ and $B = (x_B, y_B)$, define the *grid ellipse* with foci $A = (x_A, y_A)$ and $B = (x_B, y_B)$ to be the set of points $\{Q \mid d(A, Q) + d(B, Q) = s\}$. Compute the area enclosed by the grid ellipse with foci $(0, 5)$ and $(12, 0)$, passing through $(1, -1)$. [96]

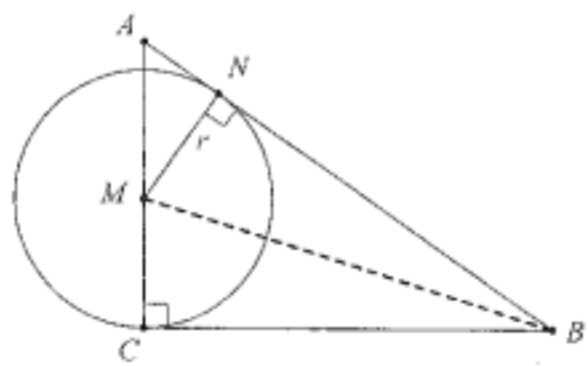
I-9. For a positive integer n, let $C(n)$ equal the number of pairs of consecutive 1's in the binary representation of n. For example, $C(183) = C(10110111_2) = 3$. Compute $C(1) + C(2) + C(3) + \cdots + C(256)$. [448]

I-10. In the complex plane, $z,\ z^2, z^3$ form, in some order, three of the vertices of a non-degenerate square. Let a and b represent the smallest and largest possible areas of the square, respectively. Compute $40(a + b)$. [425]

Team Selection Test 2
Solutions

I-1. Since $1584 \cdot x$ is a perfect cube, and $1584 = 2^4 \cdot 3^2 \cdot 11$, x must be of the form $2^{3k+2} \cdot 3^{3m+1} \cdot 11^{3n+2} \cdot r^3$, for some nonnegative integers $k, m, n, r, r > 0$. Therefore, the least positive value of x is $2^4 \cdot 3^2 \cdot 11^2 = 1452$. In order for xy to be a multiple of 1548, xy must be of the form $2^a \cdot 3^b \cdot 11^c \cdot d$, where $a \geq 4,\ b \geq 2,\ c \geq 1$, and $d \geq 1$.Therefore, y must equal $2^2 \cdot 3^1 = \mathbf{12}$.

I-2. Let N be the point of tangency of the circle with $\overline{AB}$ and let $\overline{MB}$ be drawn as below.



Since ΔBMC and ΔBMN are right triangles sharing a hypotenuse, and $\overline{MN}$ and $\overline{MC}$ are radii, $\Delta BMC \cong \Delta BMN$. Thus $BN = 12$, and $AN = 1$. Also, $\Delta ANM \sim \Delta ACB$ since they share $\angle A$, therefore $\frac{NM}{AN} = \frac{CB}{AC}$. Therefore $r = \frac{12}{5}$, so the answer is $\mathbf{17}$**.**

I-3. Let m and n be the number of men and women, respectively. Then either we choose two men and one woman, or two women and one man. Therefore:

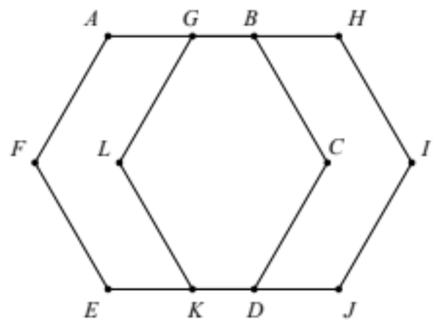
$$\binom{m}{2}\binom{w}{1} + \binom{w}{2}\binom{m}{1} = 25$$

$$\frac{m(m-1)w}{2} + \frac{w(w-1)m}{2} = 25$$

$$mw(m + w - 2) = 50.$$

Now because m, w and $m + w - 2$ are positive integer divisors of 50, and $m, w \geq 2$, we have only a few possibilities to check. If $m = 2$, then $w^2 = 25$, so $w = 5$; the case $m = 5$ is symmetric. If $m = 10$, then $w(w + 8) = 5$, which is impossible. If $m = 25$, then $w(w + 23) = 2$, which is also impossible. So $\{m, w\} = \{2{,}5\}$, and $m + w = \mathbf{7}$.

I-4.



The area of hexagon $GBCDKL$ can be computed as $[GBCDKL] = [ABCDEF] - [AGLKEF]$, and $[AGLKEF]$ can be computed by dividing concave hexagon $AGLKEF$ into two parallelograms sharing $\overline{FL}$. If $AB = a$, then the height $AE = a\sqrt{3}$, so the height of parallelogram $AGLF$ is $\frac{a\sqrt{3}}{2}$. Therefore $[AGLF] = LF \cdot \frac{a\sqrt{3}}{2}$ and $[AGLKEF] = LF \cdot a\sqrt{3}$.

On the other hand, the area of a regular hexagon of side length a is $\frac{3a^2\sqrt{3}}{2}$. Since $[GBCDKL] = \frac{1}{2}[ABCDEF]$, it follows that $[AGLKEF] = \frac{1}{2}[ABCDEF]$, and

$$LF \cdot\ a\sqrt{3} = \frac{1}{2}\left(\frac{3a^2\sqrt{3}}{2}\right) = \frac{3a^2\sqrt{3}}{4}$$

from where $LF = \frac{3}{4}a$. With $a = 24$, $LF = \mathbf{18}$.

I-5. Note that the number of digits of n is $\lfloor \log n \rfloor + 1$.
Since $100^{1000^{10,000}} = (10^2)^{1000^{10,000}}$, $X = 2 \cdot 1000^{10,000}$+1. Similarly, $Y = 3 \cdot 10{,}000^{100{,}000}$+1. With change of base formula,

$$\left\lfloor \log_X Y \frac{\log Y}{\log X} \right\rfloor \approx \frac{\log 3 + \log 10{,}000^{100{,}000}}{\log 2 + \log 1000^{10{,}000}}$$
$$= \frac{\log 3 + 100{,}000 \log 10{,}000}{\log 2 + 10{,}000 \log 1000}$$
$$= \frac{\log 3 + 100{,}000 \cdot 4}{\log 2 + 10{,}000 \cdot 3}$$
$$= \frac{400{,}000 + \log 3}{30{,}000 + \log 2}$$

When compared to $400{,}000$ and $30{,}000$, $\log 3$ and $\log 2$ are very small, and as such can be considered redundant. Therefore,

$$\lfloor \log_X Y \rfloor = \left\lfloor \frac{400{,}000}{30{,}000} \right\rfloor = \mathbf{13}.$$

I-6. The requirement that a_n be the smallest positive value of $n - a_k^2$ for $k < n$ is equivalent to determining the largest value of a_k such that $a_k^2 < n$. For $n = 3$, use either $a_1 = a_2 = 1$ to find $a_3 = 3 - 1^2 = 2$. For $n = 4$, the strict inequality eliminates a_3, so $a_4 = 4 - 1^2 = 3$, but $a_3$can be used to compute $a_5 = 5 - 2^2 = 1$. In fact, until $n = 10$, the largest allowable prior value of a_k is $a_3 = 2$, yielding the values $a_6 = 2,\ a_7 = 3,\ a_8 = 4,\ a_9 = 5$. In general, this pattern continues: from $n = m^2 + 1$ until $n = (m+1)^2$, the values of a_n increase from 1 to $2m + 1$.
Let $S_m = 1 + 2 + \cdots + (2m+1)$. Then the problem reduces to computing $S_0 + S_1 + \cdots + S_6 + 1$, because $a_{49} = 49 - 6^2$, while $a_{50} = 50 - 7^2 = 1$. $S_m = \frac{(2m+1)(2m+2)}{2} = 2m^2 + 3m + 1$, so

$$S_0 + S_1 + S_2 + S_3 + S_4 + S_5 + S_6 = 1 + 6 + 15 + 28 + 45 + 66 + 91 = 252.$$

Therefore, the desired sum is $\mathbf{253}$.

I-7. Since $|(10A + B) - (10B + A)| = 9|A - B| = k^2$, it follows that $|A - B|$ is a perfect square.
$|A - B| = 0$ yields 10 pairs of integers: $(A, B) = (0,0), (1,1), \dots, (9,9)$.
$|A - B| = 1$ yields 18 pairs: the nine $(A, B) = (0,1), (1,2), \dots, (8,9)$, and their reverses.
$|A - B| = 4$ yields 12 pairs: the six $(A, B) = (0,4), (1,5), \dots, (5,9)$, and their reverses.
$|A - B| = 9$ yields 2 pairs: $(A, B) = (0,9)$, and its reverse.
Thus, the total number of possible ordered pairs (A, B) is $\mathbf{42}$.

I-8. Let $A = (0,5)$ and $B = (12,0)$, and let $C = (1,-1)$. First compute the distance sum: $d(A, C) + d(B, C) = 19$. Notice that if $P = (x, y)$ is on the segment from $(0,-1)$ to $(12,-1)$, then $d(A, P) + d(B, P)$ is constant. This is because if $0 < x < 12$,

$$\begin{aligned} d(A,P) + d(B,P) &= |0 - x| + |5 - (-1)| + |12 - x| + |0 - (-1)| \\ &= x + 6 + (12 - x) + 1 \\ &= 19. \end{aligned}$$

Similarly, $d(A, P) + d(P, B) = 19$ whenever P is on the segment from $(0,6)$ to $(12,6)$. If P is on the segment from $(13,0)$ to $(13,5)$, then $P's$ coordinates are $(13, y)$, with $0 \le y \le 5$, and thus

$$\begin{aligned} d(A,P) + d(B,P) &= |0 - 13| + |5 - y| + |12 - 13| + |0 - y| \\ &= 13 + (5 - y) + 1 + y \\ &= 19. \end{aligned}$$

Similarly, $d(A, P) + d(P, B) = 19$ whenever P is on the segment from $(-1,0)$ to $(-1,5)$.
Finally, if P is on the segment from $(12,-1)$ to $(13,0)$, then $d(A, P) + d(B, P)$ is constant:

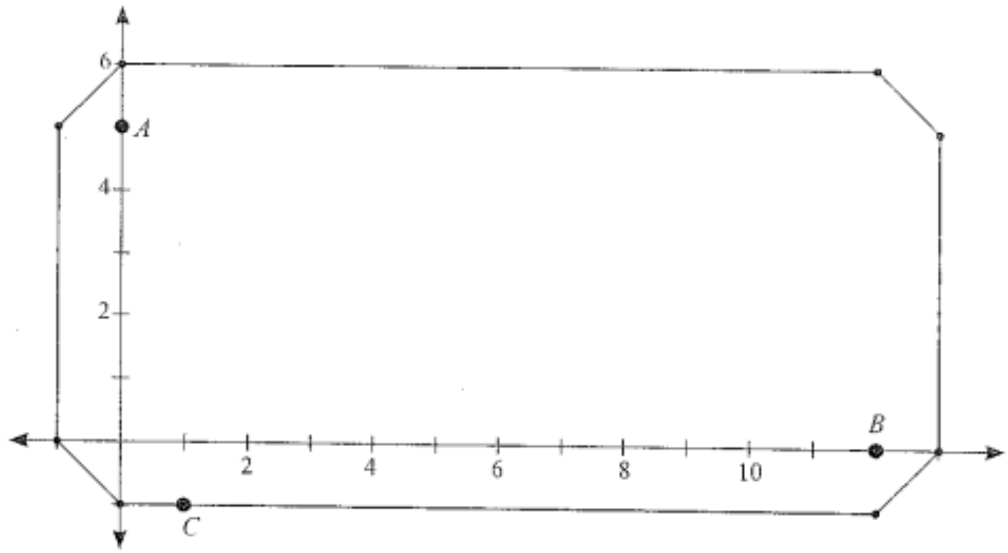
$$\begin{aligned} d(A,P) + d(B,P) &= |0 - x| + |5 - y| + |12 - x| + |0 - y| \\ &= x + (5 - y) + (x - 12) + (-y) \\ &= 2x - 2y - 7, \end{aligned}$$

and since the line segment has equation $x - y = 13$, this expression reduces to

$$d(A,P) + d(B,P) = 2(x - y) - 7$$

$$= 2(13) - 7$$
$$= 19.$$

Similarly, $d(A, P) + d(B, P) = 19$ on the segments joining $(13,5)$ and $(12,6)$, $(0,6)$ and $(-1,5)$, and $(-1,0)$ to $(0,1)$. The shape of the "ellipse" is given below.



The simplest way to compute the polygon's area is to subtract the areas of the four corner triangles from that of the enclosing rectangle. The enclosing rectangle's area is $14 \cdot 7 = 98$, while each triangle area is $\frac{1}{2}$. Therefore, the area is $98 - 4 \cdot \frac{1}{2} = \mathbf{96}$.

I-9. Let's group values of n according to the number of bits in their binary representations: Let B_n be the set of n-bit integers, and let $C_n = \sum_{k \in B_n} C(k)$ be the sum of the C-values for all

Bits	$C(n)$ values	Total
1	$C(1_2) = 0$	0
2	$C(10_2) = 0$ $C(11_2) = 1$	1
3	$C(100_2) = 0 \quad C(101_2) = 0$ $C(110_2) = 1 \quad C(111_2) = 2$	3
4	$C(1000_2) = 0 \quad C(1001_2) = 0 \quad C(1100_2) = 1 \quad C(1101_2) = 1$ $C(1010_2) = 0 \quad C(1011_2) = 1 \quad C(1110_2) = 2 \quad C(1111_2) = 3$	8

n-bit integers. Observe that the integers in B_{n+1} can be obtained by appending a 1 or a 0 to the integers in B_n. Appending a bit does not change the number of consecutive 1's in the previous (left) bits, but each number in B_n generates two different numbers in B_{n+1}. Thus, c_{n+1} equals $2c_n$ plus the number of new 11 pairs. Appending a 1 will create a new pair of consecutive 1's in and only in numbers that previously terminated in 1. The number of such numbers is half the number of elements in B_n. Because there are 2^{n-1} numbers in B_n, there are 2^{n-2} additional pairs of consecutive 1's among elements in B_{n+1}. Thus for $n \geq 2$, the sequence satisfies the recurrence relation:

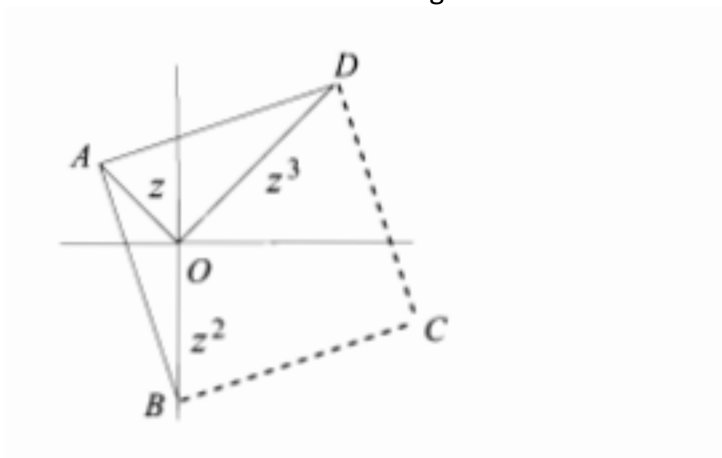
$$c_{n+1} = 2c_n + 2^{n-2}.$$

Therefore,

$$c_5 = 2c_4 + 2^{4-2} = 20$$
$$c_6 = 2c_5 + 2^{5-2} = 48$$
$$c_7 = 2c_6 + 2^{6-2} = 112$$
$$c_8 = 2c_7 + 2^{7-2} = 256$$

Since $C(256) = 0$, the desired sum is $c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 + c_8 = \mathbf{448}$.

I-10. Assume that a solution would look like the diagram below.



If we think of z, z^2 and z^3 as vectors in the complex plane, then $z^2 - z = \overline{AB}$ and $z^3 - z = \overline{AD}$. $\overline{AB}$ and $\overline{AD}$ must have the same magnitude and that magnitude equals the side of the square. Also, since multiplication by i rotates by 90°, $(\overline{AB})i = \overline{AD}$ and $(\overline{AD})(-i) = \overline{AB}$. Thus, the quotient of the two vectors equals either i or $-i$. These considerations underlie the following solution:

We will consider three cases, depending on which element of $\{z, z^2, z^3\}$ is between the other two in the square. In each case we will find possible values for z by setting the quotient of the adjacent sides, expressed as complex vectors, equal $\pm i$. That means that as vectors, the ratio of the side lengths is $|\pm i|$ and the angle between them is $\arg(\pm i) = \pm 90°$.

i) Let z be between z^2 and z^3. Then $\frac{z^3 - z}{z^2 - z} = \pm i \ \rightarrow \ z + 1 = \pm i \ \rightarrow z = -1 \pm i$. If $z = -1 + i$, $z^2 = -2i$ giving $|z^2 - z| = \sqrt{10}$, so the area is 10. If $z = -1 - i$, the area is also 10.

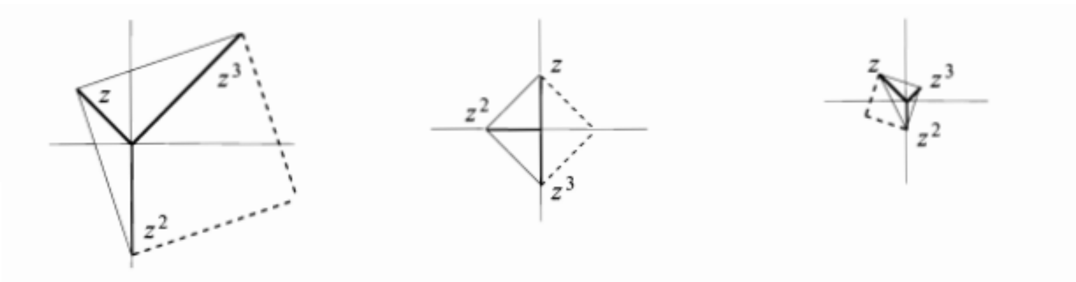
ii) Let z^2 be between z and z^3. Then $\frac{z^3 - z^2}{z^2 - z} = \pm i \ \rightarrow \ z = \pm i$. In this case $z^2 = -1$ and so $|z^2 - z| = |-1 \pm i| = \sqrt{2}$. The area of the square is 2.

iii) Finally, let z^3 lie between z and z^2. Then $\frac{z^3 - z}{z^3 - z^2} = \pm i \ \rightarrow \ \frac{z(z^2-1)}{z^2(z-1)} = \pm i \ \rightarrow \frac{z+1}{z} = \pm i$
$\frac{1}{z} = -1 \pm i \ \rightarrow \ z = -\frac{1}{2} \mp \frac{1}{2} i$. This gives $z^2 = \pm\frac{1}{2} i$, so the length of the diagonal of

the square equals $|z^2 - z| = \left|\frac{1}{2}i - \left(-\frac{1}{2} - \frac{1}{2}i\right)\right| = \left|\frac{1}{2} + i\right| = \sqrt{\frac{1}{4} + \frac{4}{4}} = \frac{\sqrt{5}}{2}$. The area is then $\left(\frac{\sqrt{5}}{2\sqrt{2}}\right)^2 = \frac{5}{8}$.

The minimum area is $\frac{5}{8}$, and the maximum area is 10, so the answer is $40\left(\frac{5}{8} + 10\right) = \mathbf{425}$.

Graphs of examples of solutions to cases (i), (ii), and (iii) respectively are shown below (left to right).



4 Power Question 2015: The Power of Riffles

Instructions: The power question is worth 50 points; each part's point value is given in brackets next to the part. To receive full credit, the presentation must be legible, orderly, clear, and concise. If a problem says "list" or "compute," you need not justify your answer. If a problem says "determine," "find," or "show," then you must show your work or explain your reasoning to receive full credit, although such explanations do not have to be lengthy. If a problem says "justify" or "prove," then you must prove your answer rigorously. Even if not proved, earlier numbered items may be used in solutions to later numbered items, but not vice versa. Pages submitted for credit should be NUMBERED IN CONSECUTIVE ORDER AT THE TOP OF EACH PAGE in what your team considers to be proper sequential order. PLEASE WRITE ON ONLY ONE SIDE OF THE ANSWER PAPERS. Put the TEAM NUMBER (not the team name) on the cover sheet used as the first page of the papers submitted. Do not identify the team in any other way.

This power question concerns sequences of positive integers, which will be written as $\langle a_1, a_2, \ldots \rangle$, or simply $\langle a_n \rangle$. In all cases, the first term of the sequence will have index 1, that is, no term will be denoted a_0.

Throughout this power event, the word "sequence" is equivalent to "positive integer sequence". Sequences may not contain non-positive or non-integer values!

A sequence such as $\langle 71, 54, 37, 20, 3, 20, 3, 20, 3, 20, 3, 20, 3, \ldots \rangle$ is called *periodic*; in this case, the period of the sequence is 2, and the periodicity begins at the fourth term. Formally, a sequence will be called periodic if there exists a positive integer p and a positive integer s such that $a_n = a_{n+p}$ for all $n \geq s$; the *period of the sequence* is the least such value of p, and the *beginning of periodicity of the sequence* is the least such value of s.

The following are four examples of sequences:

1. A *constant sequence* such as $\langle 2, 2, 2, \ldots \rangle$ is the sequence all of whose terms are 2. The sequence might also be written as $a_n \equiv 2$.
2. The powers of two are given by $a_n = 2^{n-1}$, and begin $\langle 1, 2, 4, \ldots \rangle$.
3. The Fibonacci sequence will be denoted by $\langle F_n \rangle$; they are defined by $F_1 = F_2 = 1$ and the rule $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.
4. The *modulo sequence for n*, $\langle 1, 2, 3, \ldots, n, 1, 2, 3, \ldots, n, 1, 2, \ldots \rangle$, that is, the periodic sequence satisfying the conditions (i) that the sequence has period n, (ii) that the periodicity begins with the first term, and (iii) that the first n terms are $1, 2, 3, \ldots, n$.

As the examples suggest, a sequence can be defined by an explicit or recursive formula, and need not have any easily expressible algebraic formula.

For a given sequence $\langle a_n \rangle$, the *riffle* of $\langle a_n \rangle$, denoted $\langle a'_n \rangle$, is given by setting $a'_1 = a_1$, and for $n \geq 2$:

$$a'_n = \begin{cases} a'_{n-1} - a_n & \text{if } a_n < a'_{n-1} \\ a'_{n-1} + a_n & \text{otherwise.} \end{cases}$$

Note that each term in $\langle a'_n \rangle$ is a positive integer, and so $\langle a'_n \rangle$ is indeed a sequence.

1. Compute a'_{10} for each sequence below. [5 pts]

a. $a_n = n$

b. $a_n = n + 1$

c. $a_n = n^2$

d. $a_n = 2^{n-1}$

e. $a_n = F_n$, the Fibonacci sequence

2. Compute a'_{2015} for each sequence below. [3 pts]

a. $a_n = n$
b. $a_n = 2^{n-1}$
c. the modulo sequence for $n = 5$

3. Determine the smallest n for which $a'_n = 2015$ for each sequence below, or show that no such n exists. [7 pts]

a. $a_n = n$
b. $a_n = F_n$
c. the modulo sequence for $n = 1000$
d. $a_n = 3^n - 2^n$

4. **a**. Compute examples of two sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ with equal periods such that $\langle a'_n \rangle$ and $\langle b'_n \rangle$ have different periods, or show that no such pair of sequences exists. [2 pts]
b. Show that $\langle a_n \rangle$ is periodic if and only if $\langle a'_n \rangle$ is periodic. [3 pts]

5. Suppose that $\langle a_n \rangle$ has period p and that $\langle a'_n \rangle$ has period q. [6 pts]

a. Show that $p \leq q \leq p \cdot M$, where M is the maximum value of $\langle a'_n \rangle$.
b. Determine whether p must be a divisor of q.

6. A sequence $\langle a_n \rangle$ is *invertible* if there exists at least one sequence $\langle b_n \rangle$ for which $\langle b'_n \rangle = \langle a_n \rangle$; in that case, the sequence $\langle b_n \rangle$ is an *inverse* of $\langle a_n \rangle$. Determine whether the following sequences $\langle a_n \rangle$ are invertible. [4 pts]

a. $a_n = n$
b. $a_n = \binom{n+1}{2}$
c. $a_n = 2^n$
d. $a_n = n^m$, where $m > 1$ is a fixed integer

7. **a**. Compute an example of a sequence with period 17 that is invertible, and an example of a sequence with period 17 that is not invertible. [2 pts]
b. Suppose that $\langle a_n \rangle$ is periodic with period 2 beginning at $n = 1$, and that $a_2 > 2a_1$. Show that $\langle a_n \rangle$ is invertible. [2 pts]
c. Let $\langle a_n \rangle$ be a strictly increasing sequence. That is, $a_n < a_{n+1}$ for all n. Suppose further that $\langle a_n \rangle$ is invertible. Prove that $a_n \geq 2^{n-1}$ for all n. [2 pts]
d. Determine the set of real numbers S for which the following statement is true: $\langle a_n \rangle$ is invertible if and only if $\frac{a_{n+1}}{a_n} \notin S$ for all $n \geq 1$. [3 pts]

8. Suppose $\langle a_n \rangle$ is invertible. Let $\langle a_n \rangle^{-1}$ denote the inverse of $\langle a_n \rangle$. More generally, if $k \geq 1$ and $\langle a_n \rangle^{-k}$ is invertible, denote its inverse by $\langle a_n \rangle^{-(k+1)}$. (It may be helpful to define $\langle a_n \rangle^0 = \langle a_n \rangle$.) Define $\langle a_n \rangle$ to be *k-invertible* if the sequences

$$\langle a_n \rangle, \quad \langle a_n \rangle^{-1}, \quad \ldots, \quad \langle a_n \rangle^{-(k-1)}$$

are all invertible.

a. Determine whether there exists a set S for which the following statement is true: $\langle a_n \rangle$ is 2-invertible if and only if $\frac{a_{n+1}}{a_n} \notin S$ for all $n \geq 1$. [4 pts]
b. Determine whether there exists a sequence $\langle a_n \rangle$ that is 2015-invertible but *not* 2016-invertible. [3 pts]
c. Determine whether there exists a sequence $\langle a_n \rangle$ that is k-invertible for all $k \geq 1$. (Such a sequence will be called *infinitely invertible*, or ∞-*invertible*.) [4 pts]

5 Solutions to Power Question

1. The first 10 terms of a'_n for each sequence are given below.

a. $a_n = n$: $\langle a'_n \rangle = \langle 1, 3, 6, 2, 7, 1, 8, 16, 7, \mathbf{17}, \ldots \rangle$

b. $a_n = n + 1$: $\langle a'_n \rangle = \langle 2, 5, 1, 6, 12, 5, 13, 4, 14, \mathbf{3}, \ldots \rangle$

c. $a_n = n^2$: $\langle a'_n \rangle = \langle 1, 5, 14, 30, 5, 41, 90, 26, 107, \mathbf{7}, \ldots \rangle$

d. $a_n = 2^{n-1}$: $\langle a'_n \rangle = \langle 1, 3, 7, 15, 31, 63, 127, 255, 511, \mathbf{1023}, \ldots \rangle$

e. $a_n = F_n$: $\langle a'_n \rangle = \langle 1, 2, 4, 1, 6, 14, 1, 22, 56, \mathbf{1}, \ldots \rangle$

2. **a**. Although no proof is required for this problem, it is useful to determine which values of n satisfy $a'_n = 1$. If $a'_k = 1$, the next three terms of the sequence will be $k + 2, 2k + 4, k + 1$—note the two consecutive increases—after which $\langle a'_n \rangle$ increases and decreases alternately. Thereafter, if a'_m and a'_{m+2} follow consecutive decreases, then $a'_{m+2} = a'_m - 1$, and similarly $a'_{m+2} = a'_m + 1$ if a'_m and a'_{m+2} follow consecutive increases. Thus, if $a'_{k+3} = k + 1$, the next value of m such that $a'_m = 1$ is $m = k + 3 + 2k = 3k + 3$.

Using this relation yields $a'_n = 1$ for $n = 1, 6, 21, 66, 201, 606, 1821$. Subsequently, $a'_{1824} = 1822$, and $a'_{1825} = 3647$. As 2015 is 190 terms later, $a'_{2015} = 3647 + \frac{1}{2} \cdot 190 = \mathbf{3742}$.

b. Use induction to show that $a'_n = 2^n - 1$. The base case is $a'_1 = a_1 = 1 = 2^1 - 1$. Assume then that $a'_n = 2^n - 1$. Then because $a_{n+1} = 2^{(n+1)-1} = 2^n > a'_n$, it follows that $a'_{n+1} = a'_n + a_n = 2 \cdot 2^n - 1 = 2^{n+1} - 1$. Hence $a'_{2015} = \mathbf{2^{2014}}$.

c. Note that $\langle a_n \rangle$ is periodic, and therefore bounded. A reasonable conjecture is that $\langle a'_n \rangle$ is periodic. Although no proof is required for this problem, it is useful to provide one now. First, though, it must be shown that if all the terms of a sequence $\langle a_n \rangle$ are at most M, then all the terms of $\langle a'_n \rangle$ are at most $2M$:

Proof: Assume towards a contradiction that there exists at least one term in $\langle a'_n \rangle$ that exceeds $2M$. Let k be the smallest index satisfying $a'_k > 2M$. Then $a'_{k-1} \leq 2M$, and so $\langle a'_n \rangle$ has an increase from a'_{k-1} to a'_k. As this is an increase, it must be the case that $a_k \geq a'_{k-1}$. Then $2a_k \geq a'_{k-1} + a_k = a'_k > 2M$, which implies $a_k > M$, a contradiction. □

Note that in general, the $2M$ bound cannot be improved. For example, the constant sequence $\langle 1, 1, 1, \ldots \rangle$ has a riffle $\langle 1, 2, 1, 2, \ldots \rangle$. Now it can be shown that if $\langle a_n \rangle$ is periodic, then $\langle a'_n \rangle$ is periodic as well.

Proof: If $\langle a_n \rangle$ is periodic, then it is bounded. Let its period be p and its maximum value be M. For any fixed offset r, the subsequence $\langle a'_{p+r}, a'_{2p+r}, a'_{3p+r}, \ldots \rangle$ must repeat a value in its first $2M + 1$ terms. Suppose the first instance of a repeated value in this subsequence is $a'_{kp+r} = a'_{lp+r}$. Because $a_{kp+r+c} = a_{lp+r+c}$ for all positive integers c, it follows by induction that $a'_{kp+r+c} = a'_{lp+r+c}$. Therefore $\langle a'_n \rangle$ is periodic, and its period is at most $(2M + 1)p$. □

Because $\langle a'_n \rangle$ is periodic with period 5, it is enough to find k, l, and r such that $a'_{5k+r} = a'_{5l+r}$. The first 20 terms of $\langle a'_n \rangle$ are given below.

1	3	6	2	7	6	4	1	5	10
9	7	4	8	3	2	4	1	5	10

The first instance of such a repeated pair of terms is $a'_7 = a'_{17} = 4$. Thus $\langle a'_n \rangle$ is periodic with period 10. From here it is straightforward to get $a'_{2015} = a'_{15} = \mathbf{3}$.

3. **a**. From the solution to **2a**, recall that if $k > 1$ and if $a'_k = 1$, then the next term to equal 1 will be a'_{3k+3}. Furthermore, it was shown that the terms $a'_{k+1}, a'_{k+2}, \ldots, a'_{3k+3}$ consist of alternating increases and decreases, with the terms coming after the increases or decreases themselves increasing or respectively decreasing by 1. That is, $a'_{k+2}, a'_{k+4}, \ldots, a'_{3k+2} = 2k+4, 2k+5, \ldots, 3k+4$; and $a'_{k+1}, a'_{k+3}, \ldots, a'_{3k+3} = k+2, k+1, \ldots, 1$.

For each set of terms $a_k, a_{k+1}, \ldots, a_{3k+3}$, consider the range of values covered by the respective increasing and decreasing subsequences:

k	Up	Down
1	$6, 7$	$3, 2, 1$
6	$16, 17, \ldots, 22$	$8, 7, \ldots, 1$
21	$46, 47, \ldots, 67$	$23, 22, \ldots, 1$
66	$136, 137, \ldots, 202$	$68, 67, \ldots, 1$
201	$406, 407, \ldots, 607$	$203, 202, \ldots, 1$
606	$1216, 1217, \ldots, 1822$	$608, 607, \ldots, 1$
1821	$3646, 3647, \ldots, 5467$	$1823, 1822, \ldots, 1$
5466	$10936, 10937, \ldots, 16402$	$5468, 5467, \ldots, 1$

It is clear then that the first time 2015 occurs in $\langle a'_n \rangle$ will be in the decreasing subsequence beginning at $a'_{5467} = 5468$. Because $5468 - 2015 = 3453$, it will take $2 \cdot 3453 = 6906$ more terms to reach 2015. That is, $2015 = a'_{5467+6906} = \boldsymbol{a'_{12373}}$.

b. In this case, $a'_n = 2015$ has no solution. Note that $\langle a'_n \rangle$ begins with $1, 2, 4, 1$. That is, $a'_4 = 1$.

Now suppose $a'_n = 1$, for some index $n > 2$. Then $a_{n+1} = F_{n+1} > 1$ and $a_{n+2} = F_{n+2} \geq F_{n+1} + 1$. Therefore $a'_{n+1} = F_{n+1} + 1$, $a'_{n+2} = F_{n+1} + 1 + F_{n+2} = F_{n+3} + 1$, and $a'_{n+3} = (F_{n+3} + 1) - F_{n+3} = 1$. That is, every third term of $\langle a'_n \rangle$ will equal 1, and the intermediate terms will each be 1 greater than a Fibonacci number.

This result applies to $\langle a'_n \rangle$ from $n = 4$ onward. Because 2014 is not a Fibonacci number, 2015 does not appear in $\langle a'_n \rangle$.

c. It can be shown by contradiction that 2015 does not appear in $\langle a'_n \rangle$. If $a'_n = 2015$, then because $a_n \leq 1000$, a'_{n-1} must be smaller than a'_n. That is, $a'_{n-1} = a'_n - a_n = 2015 - a_n \geq 1015$. But because $a'_{n-1} \geq 1015$ and $a_n \leq 1000$, it follows that $a'_n = a'_{n-1} - a_n < 1015$. Thus 2015 does not appear in $\langle a'_n \rangle$.

d. The number 2015 does not appear in $\langle a'_n \rangle$. First, it will be shown that $\langle a'_n \rangle$ has no decreases, or equivalently, $a_n > a'_{n-1}$ for all n. Proceed by induction. In the base case, $a_2 = 5 > 1 = a'_1$. Then assume $a'_k = a_1 + \cdots + a_k$ for some $k \geq 1$. Note that $3^1 + 3^2 + \cdots + 3^k = \frac{1}{2}(3^{k+1} - 3)$ and $2^1 + 2^2 + \cdots + 2^k = 2^{k+1} - 2$, so $a'_k = \frac{3^{k+1}}{2} - 2^{k+1} + \frac{1}{2}$. Then $a_{k+1} - a'_k = \frac{3^{k+1}-1}{2} > 0$, so $a'_{k+1} = a'_k + a_{k+1}$, which is another increase. This completes the inductive step.

Because $\langle a'_n \rangle$ is strictly increasing, it is necessary only to compute terms until reaching one that is at least 2015. The first several terms of $\langle a'_n \rangle$ are $1, 6, 25, 90, 301, 966, 3025$. Hence 2015 does not occur in the sequence.

4. **a**. There are many such examples. Consider $\langle a_n \rangle = \langle 1, 2, 1, 2, \ldots \rangle$ and $\langle b_n \rangle = \langle 1, 3, 1, 3, \ldots \rangle$: their respective riffles, with periods 4 and 6, are shown below.

$$\langle a'_n \rangle = \langle 1, 3, \mathbf{2}, 4, 3, 1, \mathbf{2}, 4, 3, 1, \ldots \rangle$$
$$\langle b'_n \rangle = \langle \mathbf{1}, 4, 3, 6, 5, 2, \mathbf{1}, 4, 3, 6, \ldots \rangle$$

b. The forward direction of the proof was shown earlier as a part of the solution to **2c**, so now consider the reverse direction of the proof.

Suppose that $\langle a'_n \rangle$ is periodic with period p. Let r be an integer such that $a'_s = a'_{s+p}$ for all $s \geq r$. Because $a_n = |a'_n - a'_{n-1}|$ for all $n > 1$, it follows that $a_{s+p} = |a'_{s+p} - a'_{s+p-1}| = |a'_{s+2p} - a'_{s+2p-1}| = a_{s+2p}$ for all $s \geq r$. This establishes the periodicity of $\langle a_n \rangle$, and completes the proof. □

5. **a**. By assumption, the sequence $\langle a_n \rangle$ has period p, the riffle $\langle a'_n \rangle$ has period q, and $\max_n a'_n = M$. Let the periodicity of $\langle a'_n \rangle$ begin at r, that is, for all $k \geq r$, $a'_k = a'_{k+q}$. Then

$$a_{k+1} = \left|a'_{k+1} - a'_k\right| = \left|a'_{k+1+q} - a'_{k+q}\right| = a_{k+1+q}$$

for all such k, and so $\langle a_n \rangle$ is periodic with period at most q. That is, $p \leq q$.

Let s be an integer for which $a_k = a_{k+p}$ for all $k \geq s$. Consider the $M+1$ values $a'_s, a'_{s+p}, \ldots, a'_{s+Mp}$. These are $M+1$ terms, all of which are values between 1 and M. Hence there must exist integers u and v such that $0 \leq u < v \leq M$ and $a'_{s+up} = a'_{s+vp}$. It can be shown via induction that $\langle a'_n \rangle$ is periodic from a'_{s+up} onwards. Assume $a'_{s+up+z} = a'_{s+vp+z}$ for some $z \geq 0$. (The case $z = 0$ has already been established.) Because $a'_{s+up+z} = a'_{s+vp+z}$ and $a_{s+up+z+1} = a_{s+vp+z+1}$, it follows that $a'_{s+up+z+1} = a'_{s+vp+z+1}$, concluding the inductive step. Thus $\langle a'_n \rangle$ is periodic with period at most $(v-u)p \leq Mp$.

b. Show that q is a multiple of p by contradiction. If $p = 1$, then q must be a multiple of p. So assume that $p > 1$. In the proof that follows, the essential idea is that if q is *not* a multiple of p, then $\langle a_n \rangle$ must be periodic with a period of $c = \gcd(p, q)$.

Assume towards a contradiction that q is not a multiple of p. Let $c = \gcd(p, q)$, and let $d = \frac{p}{c}$. Note that $c < p$, and so $d > 1$.

Let r be an integer for which $a_{r+k} = a_{r+k+p}$ for all $k \geq 0$. Then for *at least one* integer i with $1 \leq i \leq c$, the terms $a_{r+i}, a_{r+i+c}, \ldots, a_{r+i+(d-1)c}$ must contain *at least two* different values. (If this weren't the case, then $\langle a_n \rangle$ would have period at most c, which is a contradiction.)

So let x, y be integers, with $0 \leq x < y < d$ for which $a_{r+i+xc} \neq a_{r+i+yc}$. Because $c = \gcd(p, q)$, $\frac{p}{c}$ and $\frac{q}{c}$ are relatively prime. Hence there exists an integer $m > 0$ such that $m \cdot \frac{q}{c} \equiv 1 \bmod \frac{p}{c}$. Thus $m(y-x)q \equiv (y-x)c \bmod p$. Then

$$\begin{aligned} a_{r+i+xc} &= |a'_{r+i+xc} - a'_{r+i+xc-1}| \\ &= |a'_{r+i+xc+m(y-x)q} - a'_{r+i+xc+m(y-x)q-1}| \\ &= a_{r+i+xc+m(y-x)q}. \end{aligned}$$

But $r + i + xc + m(y-x)q$ is no less than r and is congruent to $r + i + yc \bmod p$, so

$$a_{r+i+xc+m(y-x)q} = a_{r+i+yc} \neq a_{r+i+xc}.$$

This result contradicts the existence of two distinct values within one of the $\frac{p}{c}$ subsequences of $\langle a_n \rangle$, which would imply that $\langle a_n \rangle$ is periodic with period at most c.

6. **a**. The sequence is not invertible. Note that if $\langle b_n \rangle$ is an inverse of $\langle a_n \rangle$, then $b_3 = |a_3 - a_2| = 1$. But then it follows that $a_3 = b'_3 = b'_2 - b_3 = 2 - 1 = 1 \neq a_3$.

b. The sequence is not invertible. The first few terms of $\langle a_n \rangle$ are $1, 3, 6, 10, 15$. So if $\langle b_n \rangle$ is an inverse of $\langle a_n \rangle$, it follows that $b_4 = |a_4 - a_3| = 4$. But then $a_4 = b'_4 = b'_3 - b_4 = 6 - 4 = 2$.

c. The sequence is invertible, with inverse $\langle b_n \rangle$, where $b_1 = 1$, and $b_n = 2^{n-2}$ for $n > 1$. It is straightforward to show by induction that $\langle b'_n \rangle = \langle a_n \rangle$.

d. This sequence is not invertible for any value of m. As with the earlier examples of sequences that were not invertible, the key is to find a pair of terms a_n, a_{n+1} for which $\frac{a_{n+1}}{a_n} < 2$. Because $\sqrt[m]{2} > 1$, there exists an integer k for which $(\frac{k+1}{k})^m < 2$. So suppose there exists a sequence $\langle b_n \rangle$ for which $b'_1, b'_2, \ldots, b'_k = a_1, a_2, \ldots, a_k$. Because $(\frac{k+1}{k})^m < 2$, it follows that $a_{k+1} < 2a_k$. Therefore $b_{k+1} = |a_{k+1} - a_k| = a_{k+1} - a_k$, which is less than $a_k = b'_k$. Thus $b'_{k+1} = b'_k - b_{k+1} = 2a_k - a_{k+1}$, which is less than a_{k+1}.

7. **a**. The periodic sequence $\langle 17, 16, 15, \ldots, 1, 17, 16, \ldots \rangle$ is invertible, whereas the periodic sequence $\langle 1, 2, 3, \ldots, 17, 1, 2, \ldots \rangle$ is not invertible.

b. Let $\langle b_n \rangle$ be defined as follows: $b_1 = a_1$, and for $n > 1$, $b_n = a_2 - a_1$. Then $b_2 > b_1$, and it follows that $b'_1 = b_1 = a_1$, $b'_2 = b_1 + b_2 = a_2$, and $b'_3 = b'_2 - b_3 = a_1$. Because $b'_1 = b'_3$ and $\langle b_n \rangle$ is constant for $n \geq 2$, $\langle b'_n \rangle$ is periodic from $n \geq 2$ onwards. Thus $\langle b_n \rangle$ is an inverse of $\langle a_n \rangle$.

c. Proceed by induction on n. For the base case, $a_1 \geq 1 = 2^{1-1}$. Suppose that $\langle b'_n \rangle = \langle a_n \rangle$, and assume that $a_n \geq 2^{n-1}$. Then $a_{n+1} \geq a_n$ implies that $b_{n+1} \geq a_n = 2^{n-1}$, so $a_{n+1} = a_n + b_{n+1} \geq 2^{n-1} + 2^{n-1} = 2^n$.

d. Given a sequence $\langle a_n \rangle$ of positive integers, define the *ruffle* sequence $\langle \hat{a}_n \rangle$ as follows: let $\hat{a}_1 = a_1$, and for $n > 1$ let $\hat{a}_n = |a_n - a_{n-1}|$.

Claim: If $\langle a_n \rangle$ is invertible, it can have only one inverse, namely $\langle \hat{a}_n \rangle$.

Proof of Claim: Suppose there exists a sequence $\langle b_n \rangle$ for which $\langle b'_n \rangle = \langle a_n \rangle$. Then $b_1 = a_1$, and for $n > 1$, $b_n = |a_n - a_{n-1}|$. That is, $\langle b_n \rangle = \langle \hat{a}_n \rangle$. □

Note the restricted nature of the claim: it does *not* guarantee that $\langle (\hat{a})'_n \rangle = \langle a_n \rangle$. For example, the ruffle of $\langle 1, 2, 3, 4, \ldots \rangle$ is the constant sequence $\langle 1, 1, 1, 1, \ldots \rangle$, but the riffle of that constant sequence is the period-2 sequence $\langle 1, 2, 1, 2, \ldots \rangle$. The reason the claim appears to generate an extraneous inverse to $\langle 1, 2, 1, 2, \ldots \rangle$ is that this sequence doesn't have an inverse, which was one of the hypotheses of the claim!

Now consider the question at hand:

Stronger Claim: A sequence $\langle a_n \rangle$ is invertible if and only if $\frac{a_{n+1}}{a_n} \notin [1, 2)$ for all $n \geq 1$.

Proof of Stronger Claim: Suppose $\langle a_n \rangle$ is a positive integer sequence such that $\frac{a_{n+1}}{a_n} \notin [1, 2)$ for all $n \geq 1$. Then $\hat{a}_n > 0$ for all $n \geq 1$. It is true that $\hat{a}_1 = (\hat{a})'_1 = a_1$. Proceed by induction to show that the ruffle sequence $\langle \hat{a}_n \rangle$ is the inverse of $\langle a_n \rangle$:

Suppose that $(\hat{a})'_k = a_k$ for some $k \geq 1$. Then $\hat{a}_{k+1} = |a_{k+1} - a_k|$. If $a_{k+1} < a_k$, then $\hat{a}_{k+1} = a_k - a_{k+1} < a_k = (\hat{a})'_k$, and so $(\hat{a})'_{k+1} = (\hat{a})'_k - \hat{a}_{k+1} = a_k - (a_k - a_{k+1}) = a_{k+1}$. If $a_{k+1} > a_k$, then $\hat{a}_{k+1} = a_{k+1} - a_k \geq a_k = (\hat{a})'_k$, and so $(\hat{a})'_{k+1} = (\hat{a})'_k + \hat{a}_{k+1} = a_k + (a_{k+1} - a_k) = a_{k+1}$. This completes the inductive step. (And the earlier claim entails that there are no other inverses of $\langle a_n \rangle$.)

To show that the condition on S is necessary, use proof by contradiction. Suppose k is the smallest integer for which $\frac{a_{k+1}}{a_k} \in [1, 2)$. By the earlier induction, $(\hat{a})'_k = a_k$. Then $\hat{a}_{k+1} = a_{k+1} - a_k < a_k = (\hat{a})'_k$, and so $(\hat{a})'_{k+1} = (\hat{a})'_k - \hat{a}_{k+1} = a_k - (a_{k+1} - a_k) = 2a_k - a_{k+1}$, which is smaller than a_k and therefore cannot equal a_{k+1}. □

8. **a**. No such set S exists. For a given sequence $\langle a_n \rangle^k$, let R^k_a be the set of values $\{\frac{a_2}{a_1}, \frac{a_3}{a_2}, \frac{a_4}{a_3}, \ldots\}$. Note that earlier it was shown that $\langle a_n \rangle^0$ is invertible if and only if $R^0_a \cap [1, 2) = \varnothing$.

Let $\langle b_n \rangle^0$ be the periodic sequence $\langle 1, 3, 1, 3, 1, 3, \ldots \rangle$. Then $R^0_b = \{\frac{1}{3}, 3\}$. This sequence is invertible, and $\langle b_n\rangle^{-1} = \langle\hat{b}_n\rangle = \langle 1, 2, 2, 2, 2, 2, \ldots\rangle$. Clearly, $\langle b_n\rangle^{-1}$ is not invertible, because it has repeated values.

Now let $\langle c_n\rangle^0$ be the sequence whose first few terms are $1, 3, 9, 30, 10$, and whose subsequent terms are $c_n = 10^{n-4}$ for $n \neq 6$. That is, the first eight terms are $1, 3, 9, 30, 10, 100, 1000, 10000$. Then $R_c^0 = \{3, \frac{10}{3}, \frac{1}{3}, 10\}$, and it follows that $R_c^0 \cap [1, 2) = \varnothing$. Thus $\langle c_n\rangle^0$ is invertible. The first several terms of $\langle c_n\rangle^{-1}$ are $2, 6, 21, 20, 90, 900, 9000$. $R_c^{-1} = \{3, \frac{7}{3}, \frac{20}{21}, \frac{9}{2}, 10\}$. It follows that $R_c^{-1} \cap [1, 2) = \varnothing$, so $\langle c_n\rangle^0$ is indeed 2-invertible.

On the other hand, $R_b^0 \subset R_c^0$. In order for the original claim to be true, either $\frac{1}{3}$ or 3 must be an element of S. But both these values are in R_c^0, and $\langle c_n\rangle^0$ is 2-invertible. Therefore the claim cannot be true, as no such set S satisfies the condition.

b. Such a sequence exists. For an integer $k \geq 1$ and sequence $\langle a_n\rangle$, let $\langle a_n\rangle^k$ denote the kth riffle of $\langle a_n\rangle$. That is, $\langle a_n\rangle^{k+1}$ is the riffle of $\langle a_n\rangle^k$. (And as before, let $\langle a_n\rangle^0 = \langle a_n\rangle$.)

Consider the kth riffle of the sequence $a_n = 2^{n-1}$. Note that $\langle a_n\rangle$ is invertible, because $\frac{a_{n+1}}{a_n} = 2$ for all n. However, because the first two terms of $\langle a_n\rangle^{-1}$ are $1, 1$, $\langle a_n\rangle$ is not 2-invertible.

Claim: Let $\langle b_n\rangle$ be the kth riffle of $\langle a_n\rangle$, where $k \geq 1$. Then $\langle b_n\rangle$ is strictly increasing, with $\frac{b_{n+1}}{b_n} > 2$ for all n.

Proof of Claim: The claim is true for $k = 1$, as $\langle a_n\rangle^1 = \langle 1, 3, 7, 15, 31, \ldots\rangle$, i.e., the sequence $2^n - 1$. So assume the claim is true for some positive integer k, and let $\langle c_n\rangle = \langle b'_n\rangle = \langle a_n\rangle^{k+1}$. First show that $\langle c_n\rangle$ is increasing. Proceed by induction. For the base case, $c_1 = b_1$, and because $b_2 > b_1$, it follows that $c_2 = b_1 + b_2$. Assume (further) now that $\langle c_n\rangle$ is increasing for $c_1, c_2, \ldots, c_j$. Note that $b_j < \frac{b_{j+1}}{2}, b_{j-1} < \frac{b_{j+1}}{4}, \ldots, b_1 < \frac{b_{j+1}}{2^j}$, and so $c_j = b_1 + b_2 + \cdots + b_j < b_{j+1}(2 - \frac{1}{2^j})$. Then because $\frac{b_{j+1}}{b_j} > 2$, $\langle c_n\rangle$ increases at c_{j+1}. This completes the (inner) inductive step, demonstrating that $\langle c_n\rangle$ is increasing. □

It remains to complete the outer induction; that is, that the ratios of consecutive terms are decreasing but are always larger than 2. It can be shown that they are all larger than 2, by noting that

$$\begin{aligned}
\frac{c_{n+1}}{c_n} &= \frac{b_1 + b_2 + \cdots + b_n + b_{n+1}}{b_1 + b_2 + \cdots + b_n} \\
&= 1 + \frac{b_{n+1}}{b_1 + b_2 + \cdots + b_n} \\
&> 1 + \frac{b_{n+1}}{\frac{b_{n+1}}{2^n} + \frac{b_{n+1}}{2^{n-1}} + \cdots + \frac{b_{n+1}}{2^1}} \\
&> 1 + \frac{1}{1 - \frac{1}{2^n}} \\
&> 2.
\end{aligned}$$

This argument completes the proof.

(**Note:** It can furthermore be shown that $\frac{c_2}{c_1} > \frac{c_3}{c_2} > \frac{c_4}{c_3} > \cdots > 2$.)

It has thus been shown that for any positive integer k, $\langle a_n\rangle^k$ is invertible. (Its ratios are all larger than 2.) Thus $\langle a_n\rangle^{2014}$ can be inverted 2014 times to get $\langle a_n\rangle$, and one more time to get $1, 1, 2, 4, 8, \ldots$, but *cannot* be inverted once more (which would be a 2016th inversion).

This solution can easily be amended to find sequences that can be inverted k times but not $k + 1$ times.

c. No such sequence $\langle a_n \rangle$ exists. Proceed by contradiction: if a sequence $\langle a_n \rangle$ is invertible, its unique inverse is the ruffle sequence, $\langle \hat{a}_n \rangle$. So assume that $\langle a_n \rangle$ is infinitely invertible. Let a_n^{-k} denote the nth term of $\langle a_n \rangle^{-k}$, which is well-defined. Then $a_1^{-k} = a_1$ for all $k \geq 1$, and $a_n^{-k} = |a_n^{-k+1} - a_{n-1}^{-k+1}|$ for all $n > 1, k \geq 1$.

Consider the sequence $a_2, a_2^{-1}, a_2^{-2}, \ldots$. The differences between consecutive terms are all just a_1. There must exist an integer s_2 such that $a_2^{-k} < a_1$ for all $k \geq s_2$. For example, if $a_1, a_2 = 4, 17$, then $a_2, a_2^{-1}, a_2^{-2}, \ldots = 17, 13, 9, 5, 1, 3, 1, 3, 1, 3, \ldots$. That is, after some point, all the values are less than a_1.

Now consider the sequence $a_3, a_3^{-1}, a_3^{-2}, \ldots$. The differences between consecutive terms are the terms of the previous sequence. Note that at some point, all the differences are at most $a_1 - 1$, after which, there must be some $s_3 \geq 1$ for which $a_3^{-s_3} < a_1 - 1$. Then all subsequent terms are at most $a_1 - 2$.

This process can be continued to show that for any $r > 1$, there exists some integer $s_r > s_{r-1}$ for which $a_r, a_r^{-1}, a_r^{-2}, \ldots$ eventually achieves a value less than $a_1 - (r-2)$. Then all subsequent terms are at most $a_1 - (r-2)$. But this cannot continue indefinitely, as eventually, some term must equal zero, which is not allowed. Hence there does not exist an infinitely invertible sequence.

Authors' Note: The authors initially stumbled upon this topic in writing individual questions; in fact, problem **2a** was "almost" an individual question. Many other questions are suggested, such as the following:

1. Does there exist an integer $N > 1$ such that the first N terms of both $\langle a_n \rangle$ and $\langle a'_n \rangle$ are permutations of the set of integers $\{1, 2, \ldots, N\}$? If so, what are all such integers?
2. For a specific sequence $\langle a_n \rangle$, is it possible to determine which positive integers never occur in its riffle $\langle a'_n \rangle$, or which occur only finitely many times in its riffle, or which occur infinitely many times in its riffle? For example, what about the sequence $a_n = n$?

Either of these questions, or extensions of the previous questions, would make a promising topic for a mathematics research project!