Connecticut ARML Qualification Test, 2022
Solutions

1. The sum of two prime numbers is 75. Find the product of these two prime numbers.

   Solution
   The sum of two odd primes is even. However, 75 is odd. Therefore, one of these two prime numbers must be even, so it must be 2. The other number is therefore 73, and the product is $2 \cdot 73 = 146$.

2. For what positive number-base b is it true that $(23_b)^2 = 562_b$?

   Solution
   We have the equation $(2b+3)^2 = 5b^2 + 6b + 2$. This reduces to $b^2 - 6b - 7 = 0$, for which the solutions are $b = 7, -1$. Thus, $b = 7$.

3. Suppose that g is a function with $g(1) = 1$, $g(2) = 3$, and with the property that, for all integers n, $g(n) = g(n-1) - g(n-2)$. Find $g(38)$.

   Solution
   Putting $n = 3$ we see that $g(3) = g(2) - g(1) = 3 - 1 = 2$. Putting $n = 4$ we see that $g(4) = -1$. Continuing in the same way, we see that the values of $g(n)$, starting at $n = 1$, are 1, 3, 2, −1, −3, −2, 1, 3, … From this sequence we see that the values of $g(n)$ are periodic, with period 6, and $g(n) = -2$ when n is a multiple of 6. Thus, $g(36) = -2$, $g(37) = 1$, and $g(38) = 3$.

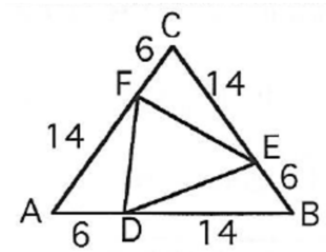
4. Find the sum of the squares of the solutions of the equation $x(|x| - 5) = -6$.

   Solution
   If $x \geq 0$ the given equation is $x(x-5) = -6$. The solutions to this equation are 2, 3.
   If $x < 0$ the given equation is $x(-x-5) = -6$, to which the solutions are −6, 1. Note that our assumption was that $x < 0$, so the solution $x = 1$ is not valid here.
   Thus, the solutions are 2, 3, −6, and the answer to the question is $2^2 + 3^2 + (-6)^2 = 49$.

5. The digits 2, 2, 3, 4, 5 will be arranged to form a five-digit positive integer, and the two 2s will not be placed next to each other. How many such five-digit integers are possible?

   Solution
   The total number of five-digit positive integers made with these digits is $5!/2 = 60$.
   To count the number of these integers that have the 2s together, consider the two 2s as a single unit. There are then four units, and the number of these integers with the 2s together is 4! = 24. So, the required number of integers is $60 - 24 = 36$.

6. In the diagram below, points D, E, F lie on line segments AB, BC, CA, respectively. The ratio of the area of triangle DEF to the area of triangle ABC is $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.



Solution

Since ABC is an equilateral triangle, each of its angles is 60°. Thus, using the law of cosines, $DE^2 = EF^2 = FD^2 = 6^2 + 14^2 - 2 \cdot 6 \cdot 14 \cos 60° = 148$. Each side of triangle ABC has length 20. Therefore, the required ratio is $148/20^2 = 37/100$, and the answer is 137.

7. Compute $99^5 + 5 \cdot 99^4 + 10 \cdot 99^3 + 10 \cdot 99^2 + 5 \cdot 99$.

Solution

Recall that $(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.
Therefore, putting $a = 99$, $b = 1$,
$99^5 + 5 \cdot 99^4 + 10 \cdot 99^3 + 10 \cdot 99^2 + 5 \cdot 99 + 1 = (99 + 1)^5 = 100^5 = 10^{10}$.
Thus, the given expression is equal to $10^{10} - 1 = 9{,}999{,}999{,}999$

8. The expression $x^4 - 11x^2 + 49$ can be factored uniquely in the form $(x^2 + px + q)(x^2 + rx + s)$, where p, q, r, s are integers. Find $p^2 + q^2 + r^2 + s^2$.

Solution

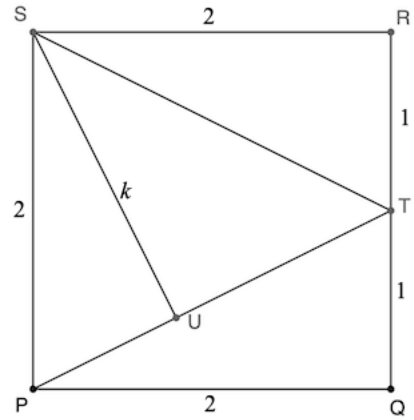
$x^4 - 11x^2 + 49 = (x^2 + 7)^2 - 25x^2 = \big((x^2 + 7) + 5x\big)\big((x^2 + 7) - 5x\big)$
$= (x^2 + 5x + 7)(x^2 - 5x + 7)$.
So, the answer is $5^2 + 7^2 + (-5)^2 + 7^2 = 148$.

9. Evaluate: $\log_2(3) \cdot \log_3(4) \cdot \log_4(5) \cdot \cdots \cdot \log_{31}(32)$.

Solution

Using the change-of-base formula, $\log_2(3) \cdot \log_3(4) \cdot \log_4(5) \cdot \cdots \cdot \log_{31}(32)$
$= \log_2 3 \cdot \frac{\log_2 4}{\log_2 3} \cdot \frac{\log_2 5}{\log_2 4} \cdot \cdots \cdot \frac{\log_2 32}{\log_2 31} = \log_2 32 = 5$.

10. In square *PQRS*, *T* is the midpoint of side *QR*. The line through *S* perpendicular to line *PT* intersects line *PT* at *U*. If *PQ* = 2, the distance *SU* is $\frac{a\sqrt{b}}{c}$, where *a*, *b*, *c* are integers, *a* and *c* are relatively prime, and *b* is not divisible by the square of any prime number. Find $a + b + c$.



Solution

We will write the area of triangle *PTS* in two ways.

Area of triangle *PTS* = $\frac{1}{2}(PT)k = \frac{1}{2} \cdot \sqrt{5} \cdot k$.

Also, area of triangle *PTS* = (Area *PQRS*) – (Area *RST*) – (Area *PQT*) = 4 – 1 – 1 = 2.

Therefore, $\frac{1}{2} \cdot \sqrt{5} \cdot k = 2$, which tells us that $k = \frac{4\sqrt{5}}{5}$. Thus, the answer is $4 + 5 + 5 = 14$.

11. Find the sum of the squares of all values of the constant *p* such that the vertex of the parabola $y = px^2 + (5p + 3)x + (6p + 5)$ lies on the *x*-axis.

Solution

We need the equation $px^2 + (5p + 3)x + (6p + 5) = 0$ to have a repeated solution. The equation $ax^2 + bx + c = 0$ has a repeated solution if and only if $b^2 - 4ac = 0$. Therefore, we need $(5p + 3)^2 - 4p(6p + 5) = 0$. The solutions to this equation are $p = -1, -9$. Therefore, the answer is $(-1)^2 + (-9)^2 = 82$.

12. Let *a* and *b* be positive integers, and suppose that the real part of $(a + bi)^3$ is −9. Find $a + b$.

Solution

$(a + bi)^3 = a^3 + 3a^2(bi) + 3a(bi)^2 + (bi)^3$.

So, the real part of $(a + bi)^3$ is $a^3 + 3a(bi)^2 = a^3 - 3ab^2$.

So, here we have $a^3 - 3ab^2 = -9$.

Therefore, $a(a^2 - 3b^2) = -9$. (1)

Thus, *a* must be a positive factor of −9. So, *a* = 1, 3, or 9.

If $a = 1$, then, by (1), $a^2 - 3b^2 = -9$, so $1 - 3b^2 = -9$. There is no positive integer value of *b* that satisfies this equation.

If $a = 3$, then, by (1), $a^2 - 3b^2 = -3$, so $9 - 3b^2 = -3$. From this we get $b = 2$.

If $a = 9$, then, by (1), $a^2 - 3b^2 = -1$, so $81 - 3b^2 = -1$. There is no positive integer value of *b* that satisfies this equation.

Thus, $a = 3$ and $b = 2$, and the answer is $2 + 3 = 5$.

13. Let $S = \{1, 2, 3, 4, \dots, 19, 20\}$. A subset of S consisting of m consecutive integers will be selected, where $m \geq 1$. Find the number of possible such subsets.

Solution
Let N_m be the number of subsets of m consecutive integers from S. $N_1 = 20,\ N_2 = 19,\ \dots,$ $N_{20} = 1$. So, the required number of subsets is $1 + 2 + \cdots + 20$. This is an arithmetic series with $a = 1,\ d = 1,$ and $n = 20$. So $1 + 2 + \cdots + 20 = \frac{20}{2}(2 \cdot 1 + 19 \cdot 1) = 210$.

14. Let g be that function such that, for all values of x, $g(x) + 3g(6 - x) = x^2 + 1$. Then $g(2) = \frac{a}{b}$, where a and b are relatively prime integers and $b > 0$. Find $a + b$.

Solution
Put $x = 2$: $g(2) + 3g(4) = 5$
Put $x = 4$: $g(4) + 3g(2) = 17$
Solving this system of equations for $g(2)$ we get $g(2) = \frac{23}{4}$. So, the answer is $23 + 4 = 27$.

15. A bag contains one blue disk and one yellow disk. A second bag contains one blue disk and three yellow disks. For either of the bags, when a disk is picked from the bag each disk in the bag is equally likely to be picked.

One disk is picked from each bag and placed on a shelf. If there is exactly one blue disk on the shelf, a further disk is picked from the bag that still contains a blue disk, and the selected disk is placed on the shelf. The probability that both blue disks are placed on the shelf during this game is $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

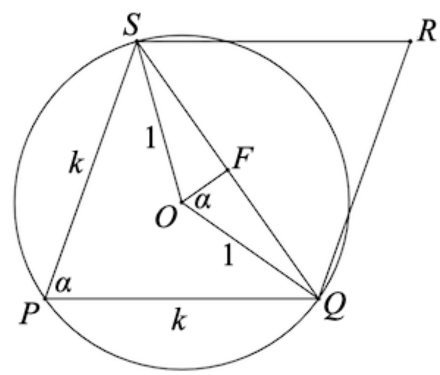
Solution
Bag 1 contains 1B, 1Y. Bag 2 contains 1B, 3Y. The required probability
= P(both Bs picked at 1st stage)
+ P(exactly one B picked at 1st stage and other B picked at 2nd stage)
$= \frac{1}{2} \cdot \frac{1}{4} +$ P(B from Bag 1, Y from Bag 2; B from Bag 2)
+ P(Y from Bag 1, B from Bag 2; B from Bag 1)
$= \frac{1}{8} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{4} \cdot 1 = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$. So, the answer is $3 + 8 = 11$.

16. Let PQR be a triangle with sides of length 3, 4, and 5. Let X be any point. The minimum possible value of $PX^2 + QX^2 + RX^2$ is $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

Solution
Place the triangle with its vertices at the points (0, 0), (3, 0), and (0, 4). Let the coordinates of X be (x, y). Then $PX^2 + QX^2 + RX^2 = x^2 + y^2 + (x-3)^2 + y^2 + x^2 + (y-4)^2$
$= 3x^2 + 3y^2 - 6x - 8y + 25$. Completing the square, we see that
$PX^2 + QX^2 + RX^2 = 3(x-1)^2 + 3\left(y - \frac{4}{3}\right)^2 + \frac{50}{3}$.
The minimum value of this expression is $\frac{50}{3}$. So, the answer to the question is $50 + 3 = 53$.

17. In a rhombus *PQRS*, suppose that each side has length *k*, $\cos(\mathrm{m}\angle P) = \frac{2}{3}$, and the radius of the circle that passes through the points *P*, *Q*, and *S* is 1. Then $k^2 = \frac{a}{b}$, where *a* and *b* are relatively prime positive integers. Find $a + b$.

<u>Solution</u>



Let $\mathrm{m}\angle P = \alpha$, the center of the circle be *O*, and the midpoint of segment *SQ* be *F*, as shown in the diagram. Note that $\sin\alpha = \sqrt{5}/3$.

Using circle theorems, $\mathrm{m}\angle SOQ = 2\alpha$. Since triangle *SOQ* is isosceles, *OF* is perpendicular to *SQ* and line *OF* bisects angle *SOQ*. Hence $\mathrm{m}\angle FOQ = \alpha$. Using trigonometry on triangle *OFQ* we see that $FQ = 1\sin\alpha = \sqrt{5}/3$. Hence, $SQ = 2\sqrt{5}/3$, so $SQ^2 = \frac{20}{9}$. Using the Law of Cosines,

$SQ^2 = k^2 + k^2 - 2 \cdot k \cdot k\cos\alpha = k^2\left(2 - 2\cdot\frac{2}{3}\right) = \frac{2k^2}{3}$.

Hence, $\frac{2k^2}{3} = \frac{20}{9}$, and so $k^2 = \frac{10}{3}$. So, the answer is $10 + 3 = 13$.

18. The sum of the four solutions, real and/or complex, of the equation

$$\frac{3x^2 - 3x + 4}{x^2 + 5x - 4} = \frac{x^2 - 6x - 9}{x^2 + x + 3}$$

is $(-1)^n \cdot \frac{a}{b}$, where *a* and *b* are relatively prime positive integers and *n* is 0 or 1. Find $n + a + b$.

<u>Solution</u>

The given equation reduces to $(3x^2 - 3x + 4)(x^2 + x + 3) = (x^2 - 6x - 9)(x^2 + 5x - 4)$.
This is a fourth-degree polynomial equation. Suppose that it is written in the form
$ax^4 + bx^3 + cx^2 + dx + e = 0$. Then, using Vieta's formulas, the sum of the solutions is $-b/a$.
Note that if we take all the terms of the equation to the left, we have $a = 3 - 1 = 2$, and
$b = (3 - 3) - (5 - 6) = 1$. So, the sum of the solutions is $-\frac{1}{2} = (-1)^1 \cdot \frac{1}{2}$.
Hence, the answer is $1 + 1 + 2 = 4$.

19. For how many ordered pairs (a, b) of positive integers is $15a + 22b = 2201$?

Solution
We need $15a + 22b = 2201$. Equivalently, $22b = 2201 - 15a$.
So, $2201 - 15a \equiv 0 \pmod{22}$. So, $15a \equiv 2201 \pmod{22}$. So, $15a \equiv 1 \pmod{22}$.
Since 15 and 22 are relatively prime, this congruence has a unique solution (mod 22). To find this solution, we list the multiples of 15, writing the answers modulo 22. This is most easily done by adding 15 each time:
15, 8, 1, …
So $15 \cdot 3 \equiv 1 \pmod{22}$, i.e. $a \equiv 3 \pmod{22}$.
Remembering that *a* must be positive, we have $a = 3, 25, 47, \dots$.
Returning to the original equation and noting that $b \geq 1$ we see that $15a \leq 2201 - 22 = 2179$.
So $a \leq 145$. Thus, the possible values of *a* are 3, 25, 47, 69, 91, 113, 134.
There are 7 possible values of a, and therefore 7 ordered pairs (a, b) of positive integers.
(Note: If you are unfamiliar with the notation used in this solution, please look up *modular arithmetic*.)

20. Suppose that $x > 0, y > 0$, and $\log_9(x) = \log_{12} y = \log_{16}(x + y)$. Then $\frac{y}{x} = \frac{a+\sqrt{b}}{c}$, where *a*, *b*, *c* are positive integers and *a*, *c* are relatively prime. Find $a + b + c$.

Solution
Put $\log_9(x) = \log_{12} y = \log_{16}(x + y) = p$. Then, $x = 9^p$, $y = 12^p$, $x + y = 16^p$.
From the first of these equations we see that $\frac{y}{x} = \frac{12^p}{9^p} = \left(\frac{4}{3}\right)^p$.
Therefore, $\left(\frac{y}{x}\right)^2 = \frac{16^p}{9^p} = \frac{x+y}{x} = 1 + \frac{y}{x}$. Thus, if we let $\frac{y}{x} = r$, we have $r^2 = 1 + r$.
From this, we get $r = \frac{1\pm\sqrt{5}}{2}$, but *r* must be positive, and so $r = \frac{1+\sqrt{5}}{2}$.
The answer is $1 + 5 + 2 = 8$.

21. A pyramid has a square base of side-length 640. The base lies in a horizontal plane and the height of the vertex of the pyramid above the base is 1024. The pyramid is now cut by a horizontal plane at height *h* above the base. The part of the pyramid above this plane is a smaller pyramid. For how many integer values of *h* with $1 \leq h \leq 1023$ is the volume of the smaller pyramid an integer?

Solution
Let the height of the smaller pyramid be *H*, so that $h + H = 1024$. The question asked is equivalent to asking for how many integer values of *H*, $1 \leq H \leq 1023$, is the height of the smaller pyramid an integer.
Using similarity, the side length of the base of the smaller pyramid is $\frac{H}{1024} \cdot 640 = \frac{5H}{8}$.
So, the volume of the smaller pyramid is $\frac{1}{3}\left(\frac{5H}{8}\right)^2 \cdot H = \frac{25}{3}\left(\frac{H}{4}\right)^3$.
This volume is an integer if and only if *H* is a multiple of 4 and a multiple of 3; that is, if and only if *H* is a multiple of 12.
Since $\frac{1023}{12} = 85\frac{1}{4}$, the number of multiples of 12 between 1 and 1023 is 85.

22. Let α be an angle such that $\sin\alpha, \sin 2\alpha, \sin 4\alpha$ is an arithmetic sequence with nonzero common difference. Then $\cos\alpha - \cos^3\alpha = \frac{a}{b}$, where a and b are relatively prime integers and $b > 0$. Find $a + b$.

Solution
We need $\sin 2\alpha - \sin\alpha = \sin 4\alpha - \sin 2\alpha$,
i.e. $2\sin\alpha\cos\alpha - \sin\alpha = 2\sin 2\alpha\cos 2\alpha - \sin 2\alpha$,
i.e. $\sin\alpha\,(2\cos\alpha - 1) = \sin 2\alpha\,(2\cos 2\alpha - 1)$,
i.e. $\sin\alpha\,(2\cos\alpha - 1) = 2\sin\alpha\cos\alpha\,(2\cos 2\alpha - 1)$.
If $\sin\alpha$ were 0, then the common difference of the original sequence would be 0. So $\sin\alpha \neq 0$.
Thus, $2\cos\alpha - 1 = 2\cos\alpha\,(2\cos 2\alpha - 1)$,
i.e. $2\cos\alpha - 1 = 2\cos\alpha\,(4\cos^2\alpha - 3)$.
This equation reduces to $\cos\alpha - \cos^3\alpha = \frac{1}{8}$, and the answer is $1 + 8 = 9$.

23. The *period* of a sequence $a_0, a_1, a_2, a_3, \ldots$ is the smallest positive integer *k* such that $a_{n+k} = a_n$ for all $n \geq 0$. A sequence is given by $a_0 = \tan 4°$, $a_n = \frac{2a_{n-1}}{1-a_{n-1}^2}$ (for $n > 0$). Find the period of this sequence.

Solution
Note that if $a_{n-1} = \tan\theta$ then $a_n = \frac{2\tan\theta}{1-\tan^2\theta} = \tan 2\theta$.

Also, note that for any angle θ, $\tan\theta = \tan(\theta - 180°)$.

So, we can write the first few terms of the sequence as follows:
$\tan 4°, \tan 8°, \tan 16°, \tan 32°, \tan 64°, \tan 128°, \tan 76°, \tan 152°, \tan 124°, \tan 68°, \tan 136°, \tan 92°, \tan 4°, \tan 8°, \tan 16°, \ldots$
This sequence is periodic, with period = 12.

24. Find the sum of all real values of *x* that satisfy the equation $\sqrt[3]{5x + \sqrt[3]{5x + 11}} = 11$.

Solution
Note that the expression on the left-hand side of the equation represents an increasing function of *x*. Therefore, the equation has exactly one solution.
Now, let α be that number such that $\sqrt[3]{5\alpha + 11} = 11$.
Then, putting $x = \alpha$, the left-hand side of the equation is $\sqrt[3]{5\alpha + 11}$, which is equal to 11, as required.
So, the solution to the given equation is $x = \alpha$.
From $\sqrt[3]{5\alpha + 11} = 11$ we get $5\alpha + 11 = 11^3 = 1331$. So the solution is $x = \alpha = \frac{1320}{5} = 264$.

25. For how many integers n, $1 \le n \le 100$, is $x^{2n} + 1 + (x+1)^{2n}$ divisible by $x^2 + x + 1$?

Solution

In this solution we will write $\cos\theta + i\sin\theta$ as $\operatorname{cis}\theta$.

Let $P_n(x) = x^{2n} + 1 + (x+1)^{2n}$.

The solutions to the equation $x^2 + x + 1 = 0$ are $x = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i = \operatorname{cis}\left(\frac{2\pi}{3}\right), \operatorname{cis}\left(-\frac{2\pi}{3}\right)$.

Therefore, by the factor theorem, $P_n(x)$ is divisible by $x^2 + x + 1$ if and only if $P_n\left(\operatorname{cis}\frac{2\pi}{3}\right) = P_n\left(\operatorname{cis}\left(-\frac{2\pi}{3}\right)\right) = 0$.

Note, first, that $\operatorname{cis}\frac{2\pi}{3} + 1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i = \operatorname{cis}\frac{\pi}{3}$ and $\operatorname{cis}\left(-\frac{2\pi}{3}\right) + 1 = \frac{1}{2} - \frac{\sqrt{3}}{2}i = \operatorname{cis}\left(-\frac{\pi}{3}\right)$.

So, $P_n\left(\operatorname{cis}\frac{2\pi}{3}\right) = \left(\operatorname{cis}\frac{2\pi}{3}\right)^{2n} + 1 + \left(\operatorname{cis}\frac{\pi}{3}\right)^{2n} = 1 + \operatorname{cis}\frac{4n\pi}{3} + \operatorname{cis}\frac{2n\pi}{3}$.

If, for some integer k, $n = 3k$, $P_n\left(\operatorname{cis}\frac{2\pi}{3}\right) = 1 + \operatorname{cis}4k\pi + \operatorname{cis}2k\pi = 1 + 1 + 1 = 3 \ne 0$.

If $n = 3k+1$, $P_n\left(\operatorname{cis}\frac{2\pi}{3}\right) = 1 + \operatorname{cis}\left(4k\pi + \frac{4\pi}{3}\right) + \operatorname{cis}\left(2k\pi + \frac{2\pi}{3}\right) = 1 + \operatorname{cis}\frac{4\pi}{3} + \operatorname{cis}\frac{2\pi}{3}$
$= 1 + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 0$.

If $n = 3k+2$, $P_n\left(\operatorname{cis}\frac{2\pi}{3}\right) = 1 + \operatorname{cis}\left(4k\pi + \frac{8\pi}{3}\right) + \operatorname{cis}\left(2k\pi + \frac{4\pi}{3}\right) = 1 + \operatorname{cis}\frac{2\pi}{3} + \operatorname{cis}\frac{4\pi}{3} = 0$.

So, $P_n\left(\operatorname{cis}\frac{2\pi}{3}\right) = 0$ if and only if n is not a multiple of 3.

Working in a similar way we see that, likewise, $P_n\left(\operatorname{cis}\left(-\frac{2\pi}{3}\right)\right) = 0$ if and only if n is not a multiple of 3. (This is actually more easily done by noting that $\operatorname{cis}\left(-\frac{2\pi}{3}\right)$ is the complex conjugate of $\operatorname{cis}\frac{2\pi}{3}$. Since P_n has real coefficients, it follows that $P_n\left(\operatorname{cis}\left(-\frac{2\pi}{3}\right)\right)$ is the complex conjugate of $P_n\left(\operatorname{cis}\frac{2\pi}{3}\right)$, that is, the complex conjugate of zero, which is zero.)

So, the acceptable values of n are those integers between 1 and 100 inclusive that are not multiples of 3. That is, all integers between 1 and 100 inclusive except for 3, 6, 9, … , 99, and there are 33 numbers in this list. So, the answer is $100 - 33 = 67$.