**Connecticut ARML Qualification Test, 2023**
**Solutions**

1. Let $i = \sqrt{-1}$. The value of

$$\frac{1}{1 + \frac{1}{2 - i}}$$

is $a + bi$, where $a$ and $b$ are real. Find $100(a^2 + b^2)$.

Solution:

$$\frac{1}{1 + \frac{1}{2 - i}} = \frac{1}{\frac{3 - i}{2 - i}} = \frac{2 - i}{3 - i} = \frac{(2 - i)(3 + i)}{(3 - i)(3 + i)} = \frac{7 - i}{10}$$

So $a = \frac{7}{10}, b = -\frac{1}{10}, and\ 100\,(a^2 + b^2) = \boxed{50}$

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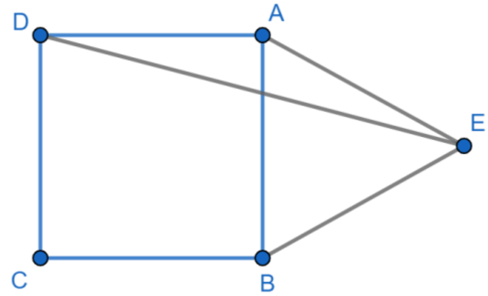
2. In the diagram below, *ABCD* is a square, *ABE* is an equilateral triangle, and point *E* lies outside square *ABCD*. Find the degree measure of $\angle CDE$.
(In this test, do not attempt to include units in your answers.)




Solution:
Based on the fact that ABCD is a square and ABE is an equilateral triangle, triangle ADE is an isosceles triangle with $\angle DAE = 90 + 60 = 150°$. Thus $\angle ADE$ is 15 degree and $\angle CDE$ is $\boxed{75}$ degrees.

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3. Evaluate

$$\left(\frac{27}{3^{\sqrt{3}}}\right)^{(3+\sqrt{3})}$$

Solution:

$(\frac{27}{3^{\sqrt{3}}})^{(3+\sqrt{3})} = (3^{(3-\sqrt{3})})^{(3+\sqrt{3})} = 3^6 = 729$

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4. Let $P(x) = ax^7 + bx^3 + cx - 7$, where $a, b, c$ are constants. Given that $P(3) = 5$, find $|P(-3)|$.

Solution:

$$P(3) = a \times 3^7 + b \times 3^3 + c \times 3 - 7 = 5$$
$$a \times 3^7 + b \times 3^3 + c \times 3 = 12$$
$$|P(-3)| = |a \times (-3)^7 + b \times (-3)^3 + c \times (-3) - 7| = |-12 - 7| = \boxed{19}$$

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5. Find the number of integer solutions to the equation $|2x + 7| + |2x - 1| = 8$.

Solution:
Case 1: If $x \geq \frac{1}{2}$, both $(2x + 7)\ and\ (2x - 1) \geq 0$, simplify the equation we get $x = \frac{1}{2}$. No integer solution
Case 2: If $-\frac{7}{2} \leq x < \frac{1}{2}$, the LHS can be simplified to $2x + 7 + 1 - 2x = 8$. This equation satisfies for any numbers in this range. Number of integer solutions is 4 ( -3, -2, -1, 0).
Case 3: If $x < -\frac{7}{2}$, likewise we can solve to get $x = -\frac{7}{2}$. No integer solutions.
So the answer is $\boxed{4}$

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6. Regular polygon *ABCDE*… has the property that $\angle ACD = 120°$. How many sides does the polygon have?

Solution:
Let the degree of the interior angle of the regular polygon be $x$. We realize that triangle ABC is an isosceles triangle because AB=BC.

$$\angle ABC = x, \angle ACB = \angle BCD - \angle ACD = x - 120, \angle BAC = \angle ACB$$
$$\therefore x + 2(x - 120) = 180. Solve\ the\ equation\ we\ get\ x = 140$$

The sum of the interior angles is $140n, and\ also\ (n - 2) \times 180$

$$140n = (n - 2) \cdot 180$$
$$n = \boxed{9}$$

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7. Let $x$ be a real number such that $\sec x - \tan x = 2$. Then $\sec x + \tan x = \frac{a}{b}$, where $a$ and $b$ are relatively prime positive integers. Find $a + b$.

Solution:

$$1 = sec^2 x - tan^2 x = (\sec x + \tan x)(\sec x - \tan x)$$
$$\therefore \sec x + \tan x = \frac{1}{2}$$
$$a + b = \boxed{3}$$

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8. Let the finite sequence $a_1, a_2, a_3, \ldots, a_{20}$ be defined by $a_1 = 72$ and $a_n = \phi(a_{n-1})$ for $2 \le n \le 20$, where $\phi(n)$ is the number of positive integer divisors of *n*. Find the sum of the twenty terms of the sequence.

<u>Solution:</u>

$$a_1 = 72 = 2^3 \times 3^2$$
$$a_2 = \phi(a_1) = 12 = 2^2 \times 3^1$$
$$a_3 = \phi(a_2) = 6 = 2^1 \times 3^1$$
$$a_4 = \phi(a_3) = 4 = 2^2$$
$$a_5 = \phi(a_4) = 3$$
$$a_6 = a_7 = \cdots = a_{20} = 2$$
$$\sum_{n=1}^{20} a_n = 72 + 12 + 6 + 4 + 3 + 2 \times 15 = \boxed{127}$$

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9. Find the number of integers *n*, with $-10 \le n \le 10$, such that *n* lies in the domain of the function *f*, where

$$f(x) = \frac{\sqrt{x^2 - 5x - 6}}{x^2 + x - 30}$$

<u>Solution:</u>

First, the expression under the square root needs to be nonnegative

$$x^2 - 5x - 6 = (x-6)(x+1) \ge 0$$
$$x \in (-\infty, -1] \cup [6, +\infty)$$

Second, the denominator cannot be zero

$$x^2 + x - 30 \ne 0$$
$$x \ne -6, 5$$

So the number of integers *n* with $-10 \le n \le 10$ such that *n* lies in the domain of $f$ is $\boxed{14}$.

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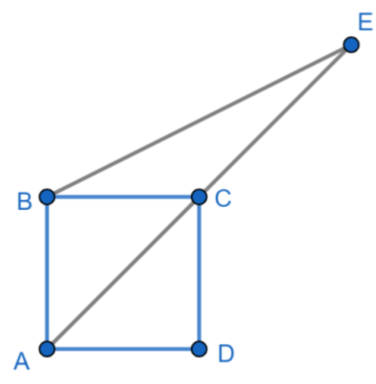
10. In the diagram below, the area of square *ABCD* is 1. Point *E* is on ray $\overrightarrow{AC}$ such that $CE = AC$. Find $BE^2$.




<u>Solution:</u>

Draw line CF$\parallel BE$ that intersects AB at point F.

Since C is the midpoint of AE and CF$\parallel BE$, BE = 2 CF = $2\sqrt{1^2 + (\frac{1}{2})^2} = \sqrt{5}$

$$BE^2 = \boxed{5}$$

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11. Find the largest solution $x$, in degrees, with $0 \leq x \leq 360°$, of the equation $\sin x \cos x = \frac{1}{4}$.

Solution:

$$\sin x \cos x = \frac{1}{4}$$
$$\sin 2x = 2\sin x \cos x = \frac{1}{2}$$
$$2x = 30, 150, 390, 510$$
$$x = 15, 75, 195, 255$$
$$\max(x) = \boxed{255}$$

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12. Let $a$, $b$, $c$ be real numbers. Find the minimum possible value of

$$3a^2 + 27b^2 + 5c^2 - 18ab - 30c + 217$$

Solution:

$$3a^2 + 27b^2 + 5c^2 - 18ab - 30c + 217$$
$$= 3a^2 - 18ab + 27b^2 + 5c^2 - 30c + 45 + 172$$
$$= 3(a - 3b)^2 + 5(c - 3)^2 + 172 \geq 172$$

So the minimum value is $\boxed{172}$.

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13. Alice and George have a fair 8-sided die with the numbers 0, 1, 2, 9, 2, 0, 1, 1 written on the faces. If Alice and George each roll the die once, the probability that Alice rolls a larger number than George does is $\frac{a}{b}$, where $a$ and $b$ are relatively prime positive integers. Find $a + b$.

Solution:
This problem can be solved by either casework or complimentary counting. Here we show a solution using complimentary counting.
The probability of both Alice and George roll number 0 is $(\frac{1}{4})^2 = \frac{1}{16}$; the probability of both Alice and George roll number 1 is $(\frac{3}{8})^2 = \frac{9}{64}$; the probability of both Alice and George roll number 2 is $(\frac{1}{4})^2 = \frac{1}{16}$; and the probability of both Alice and George roll number 9 is $(\frac{1}{8})^2 = \frac{1}{64}$. So the probability that they roll two different numbers is $1 - \left(\frac{1}{16} + \frac{9}{64} + \frac{1}{16} + \frac{1}{64}\right) = \frac{23}{32}$.
Thus the probability that Alice rolls a larger number than George does is $\frac{23}{64}$.
The answer is $\boxed{87}$.

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14. Let $x$, $y$, and $z$ be the roots of the equation $t^3 - 3t^2 + 2t - 4 = 0$.
Find $(x+1)(y+1)(z+1)$.

Solution:

$$(x+1)(y+1)(z+1) = xyz + (xy + yz + zx) + (x + y + z) + 1$$

Using Vieta's formula, we have $xyz = 4, xy + yz + zx = 2, x + y + z = 3$

The answer is $\boxed{10}$.

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15. If $\sin x + \cos x = \frac{1}{3}$, then $\sin^3 x + \cos^3 x = \frac{a}{b}$, where $a$ and $b$ are relatively prime positive integers. Find $a + b$.

Solution:

$$\sin x + \cos x = \frac{1}{3}$$

$$(\sin x + \cos x)^2 = 1 + 2\sin x \cos x = \frac{1}{9}$$

$$\sin x \cos x = -\frac{4}{9}$$

$$\sin^3 x + \cos^3 x = (\sin x + \cos x)(sin^2 x - \sin x \cos x + cos^2 x)$$

$$= \frac{1}{3}\left(1 + \frac{4}{9}\right) = \frac{13}{27}$$

The answer is $\boxed{40}$.

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16. Suppose that $x + \log_8 3$, $x + \log_4 3$, $x + \log_2 3$ are consecutive terms of a geometric sequence. Find the common ratio of the sequence.

Solution:

$Let\ a = \log_2 3$, from property of log function, we know $\log_4 3 = \frac{a}{2}, and\ \log_8 3 = \frac{a}{3}$.

Since these three numbers form a geometric sequence,

$$(x + \frac{a}{2})^2 = (x + \frac{a}{3})(x + a)$$

$$x^2 + ax + \frac{a^2}{4} = x^2 + \frac{4}{3}ax + \frac{a^2}{4}$$

$$x = -\frac{1}{4}a$$

Thus the three terms are $\frac{a}{12}, \frac{a}{4}, \frac{3a}{4}$ respectively and the common ratio is $\boxed{3}$.

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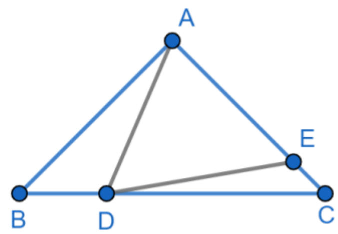
17. In the diagram below, $AB = AC$, $m\angle BAD = 30°$ and $AE = AD$. Find the degree measure of $\angle CDE$.




Solution:
Let $\angle DAE = 2x$
Since AE = AD and triangle DAE is an isosceles triangle, $\angle AED = 90 - x$.
Similarly since AB=AC and triangle ABC is an isosceles triangle, $\angle C = 75 - x$.
$\angle AED$ is an exterior angle of triangle CDE and $\angle AED = \angle CDE + \angle C$.
Thus $\angle CDE = \boxed{15}$ degree.

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18. Find the remainder when $2022^{2022}$ is divided by 5.

Solution:

$$2022^{2022} (mod\ 5) \equiv 2^{2022} (mod\ 5)$$
$$\equiv 2^2 \cdot 2^{2020} (mod\ 5) \equiv 2^2 \cdot 16^{505} (mod\ 5)$$
$$\equiv 2^2 \cdot 1\ (mod\ 5) \equiv 4\ (mod\ 5)$$

The remainder is $\boxed{4}$.

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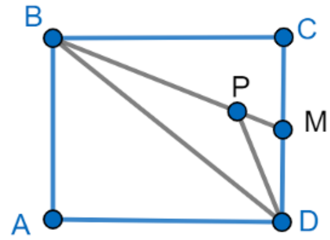
19. The diagram below shows rectangle $ABCD$ with $AB = 10$ and $BC = 12$. Let $M$ be the midpoint of side $CD$ and $P$ be the point on line segment $BM$ such that $BP = BC$. Then the area of quadrilateral $ABPD$ is $\frac{p}{q}$, where $p$ and $q$ are relatively prime positive integers. Find $p + q$.




Solution:
We can easily find that triangle BCM is a right triangle with side length of 5, 12, and 13.

$$S_{\Delta BCM} = 30$$

Since M is the midpoint of CD and triangle BCM and triangle BMD share the same height on CD,

$$S_{\Delta BDM} = 30$$

BP = BC =12, PM =1

$$\frac{S_{\Delta PMD}}{S_{\Delta BDM}} = \frac{PM}{BM} = \frac{1}{13}$$

$$S_{\Delta PMD} = \frac{30}{13}$$

$$S_{ABPD} = S_{ABCD} - S_{\Delta BCM} - S_{\Delta PMD} = 120 - 30 - \frac{30}{13} = \frac{1140}{13}$$

The answer is $\boxed{1153}$.

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20. How many integers $n$, with $1 \le n \le 100$, can be expressed as the difference of two perfect squares? (Note: 0 is a perfect square.)

Solution:
For all the odd numbers ($2k$ +1), it can be expressed as the difference of $(k+1)^2 - k^2 = 2k + 1$.
We have 50 odd numbers.
For the even numbers, if it can be expressed as the difference of two perfect squares, it has to be either two even numbers or two odd numbers. Otherwise the difference of the square is odd. $n = a^2 - b^2 = (a+b)(a-b)$, so both (a+b) and (a-b) have to be even, and n must be divided by 4. We have 25 even numbers divisible by 4.
The answer is $\boxed{75}$.

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21. Evaluate $\sqrt{97 \cdot 98 \cdot 99 \cdot 100 + 1}$

Solution:

$$\sqrt{n(n+1)(n+2)(n+3)+1} = \sqrt{n(n+3)(n+1)(n+2)+1}$$
$$= \sqrt{(n^2+3n)(n^2+3n+2)+1} = \sqrt{(n^2+3n)^2 + 2(n^2+3n)+1}$$
$$= \sqrt{(n^2+3n+1)^2} = n^2 + 3n + 1$$

Substituting $n = 97$, $\sqrt{97 \cdot 98 \cdot 99 \cdot 100 + 1} = 97(97+3) + 1 = \boxed{9701}$.

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22. Let $f(x) = \sqrt{2x+1+2\sqrt{x^2+x}}$. Evaluate

$$\frac{1}{f(1)} + \frac{1}{f(2)} + \cdots + \frac{1}{f(24)}$$

Solution:

$$f(x) = \sqrt{2x+1+2\sqrt{x^2+x}} = \sqrt{(x+1)+2\sqrt{(x+1)x}+x}$$
$$= \sqrt{(\sqrt{x+1}+\sqrt{x})^2} = \sqrt{x+1}+\sqrt{x}$$
$$\frac{1}{f(1)} + \frac{1}{f(2)} + \cdots + \frac{1}{f(24)} =$$
$$\frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{\sqrt{4}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{25}+\sqrt{24}}$$
$$= \sqrt{2}-1+\sqrt{3}-\sqrt{2}+\sqrt{4}-\sqrt{3}+\cdots+\sqrt{25}-\sqrt{24}$$
$$= \sqrt{25}-1 = \boxed{4}$$

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23. Let $f(x) = \frac{2}{4^x+2}$. The value of

$$f\left(\frac{1}{2022}\right) + f\left(\frac{2}{2022}\right) + \cdots + f\left(\frac{2021}{2022}\right)$$

is $\frac{a}{b}$, where *a* and *b* are relatively prime positive integers. Find $a+b$.

Solution:

$$f(x) = \frac{2}{4^x+2}$$
$$f(1-x) = \frac{2}{4^{(1-x)}+2} = \frac{2\cdot 4^x}{4+2\cdot 4^x} = \frac{4^x}{4^x+2}$$
$$f(x)+f(1-x) = 1$$
$$f\left(\frac{1}{2022}\right) + f\left(\frac{2}{2022}\right) + \cdots + f\left(\frac{2021}{2022}\right) =$$
$$f\left(\frac{1}{2022}\right) + f\left(\frac{2021}{2022}\right) + f\left(\frac{2}{2022}\right) + f\left(\frac{2020}{2022}\right) + \cdots + f\left(\frac{1010}{2022}\right) + f\left(\frac{1012}{2022}\right) + f\left(\frac{1011}{2022}\right) =$$
$$1010 + f\left(\frac{1}{2}\right) = 1010 + \frac{1}{2} = \frac{2021}{2}$$

The answer is $\boxed{2023}$.

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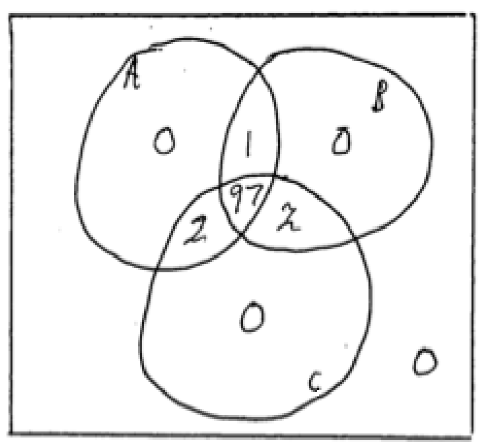
24. For any set $P$, let $|P|$ denote the number of elements in $P$, and let $n(P)$ denote the number of subsets of $P$, including the empty set and $P$ itself. Suppose, now, that $A$, $B$, and $C$ are sets for which $n(A)+n(B)+n(C)=n(A\cup B\cup C)$ and $|A|=|B|=100$. What is the minimum possible value of $|A\cap B\cap C|$ ?

<u>Solution:</u>
First recall that if the number of elements of a set $S$ is $n$, then the number of subsets of $S$ is $2^n$. Also, note that, if $2^m+2^n=2^k$, where $m$, $n$ and $k$ are positive integers, then $m=n$. (To prove this, assume that $m<n$ and show that dividing both sides by $2^m$ gives us a contradiction.)

The information in the question tells us that $2^{100}+2^{100}+2^{|C|}=2^{|A\cup B\cup C|}$, so $2^{101}+2^{|C|}=2^{|A\cup B\cup C|}$. So, by the result above, $|C|=101$ and $|A\cup B\cup C|=102$. Thus, we have $|A|=100$, $|B|=100$, $|C|=101$ and $|A\cup B\cup C|=102$. (1)

Now, we want to minimize $|A\cap B\cap C|$. Let the universal set be $A\cup B\cup C$, meaning that there's nothing outside $A\cup B\cup C$. Minimizing $|A\cap B\cap C|$ is equivalent to maximizing $|(A\cap B\cap C)^C|$. But $(A\cap B\cap C)^C=A^C\cup B^C\cup C^C$. So, we want to maximize $|A^C\cup B^C\cup C^C|$. Note that the conditions in (1) are equivalent to saying $|A^C|=2$, $|B^C|=2$, and $|C^C|=1$. Thus, the largest possible value of $|A^C\cup B^C\cup C^C|$ is $2+2+1=5$, achieved by $A^C, B^C, C^C$ being pairwise disjoint. In that case, $|(A\cap B\cap C)^C|=5$, so the smallest possible value of $|A\cap B\cap C|$ is 102 – 5 = $\boxed{97}$.

This solution is illustrated in the Venn diagram below.



25. For complex constant $c$ and real constants $p$ and $q$ there are three distinct complex values of $z$ that satisfy the equation $z^3 + cz + p(1 + qi) = 0$. Suppose that $c$, $p$, and $q$ are chosen so that all three complex roots $z$ satisfy $\frac{5}{6} \leq \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)} \leq \frac{6}{5}$, where $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ denote the real and imaginary parts of $z$, respectively. The largest possible value of $|q|$ is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

Solution:

For $z = a + bi = re^{i\theta}$, θ is called the argument of z and satisfies $\theta = tan^{-1}\frac{b}{a}$. Since b= Im(z) and a = Re(z), $\frac{Im(z)}{Re(z)} = \tan(\arg(z)) = \tan\theta$.

Let $z_1, z_2, and\ z_3$ be the three complex roots of the equation $z^3 + cz + p(1 + qi) = 0$.

By Vieta, $z_1 z_2 z_3 = -p(1 + qi) = -p - pqi$

$$\frac{Im(z_1 z_2 z_3)}{Re(z_1 z_2 z_3)} = \tan(\arg(z_1 z_2 z_3)) = -q$$

$\arg(z_1 z_2 z_3) = \theta_1 + \theta_2 + \theta_3$ and $q = -\tan(\theta_1 + \theta_2 + \theta_3)$. To maximize $|q|$, we want to make $\theta_1 + \theta_2 + \theta_3$ as close as possible to an odd integer multiple of $\frac{\pi}{2}$. Since $\frac{5}{6} \leq \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)} \leq \frac{6}{5}$, we can know that $\theta_1, \theta_2, \theta_3$ are in the range of $\frac{\pi}{6}\ and\ \frac{\pi}{3}$, thus $\theta_1 + \theta_2 + \theta_3\ is$ in the range of $\frac{\pi}{2}\ and\ \pi$. So we want to set each θ to its minimum possible value, which is $tan^{-1}\frac{5}{6}$. Thus,

$$q = -\tan(3\theta) = -\frac{3\tan\theta - tan^3\theta}{1 - 3tan^2\theta} = -\frac{3 \times \frac{5}{6} - \frac{125}{216}}{1 - 3 \times \frac{25}{36}} = \frac{415}{234}$$

The answer is $\boxed{649}$.

This answer can be verified by choosing $z_1 = e^{i\theta}$, $z_2 = 2e^{i\theta}, and\ z_3 = 3e^{i(\pi+\theta)}$.