MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS - 2023 - SOLUTIONS

Round 1: Arithmetic and Number Theory

1. Since $0.\overline{12}$ repeats with a period of 2 and $0.\overline{252}$ repeats with a period of 3, their sum will repeat with a period of 6. Thus, from $0.1212\overline{12}+0.252\overline{252}$, we obtain $0.\overline{373464}$. Then from $N=0.\overline{373464}$, we obtain $1{,}000{,}000N=373464.\overline{373464}$. Subtraction gives $999{,}999N=373464$. Note that the sum of the digits of 373,464 is divisible by 9, so divide both sides by 9, obtaining $111{,}111N=41{,}496$. Note that the sum of the digits of 41,496 is divisible by 3, so divide both sides by 3, obtaining $37{,}037N=13{,}832$. The left side factors as $37\cdot 1{,}001$ which equals $37\cdot 7\cdot 11\cdot 13$. The right side factors as $8\cdot 7\cdot 13\cdot 19$ which yields $37\cdot 11\cdot N=8\cdot 19\rightarrow 407N=152\rightarrow N=\frac{152}{407}$. Thus, $a+b=\boxed{559}$.

2. We want $\frac{7N}{4},\frac{14N}{3}$, and $\frac{21N}{5}$ to be integers. Think of N as $\frac{a}{b}$. Then we want the following to be integers: $\frac{7a}{4b},\frac{14a}{3b}$, and $\frac{21a}{5b}$. Thus, a should be divisible by 3, 4, and 5. The smallest value of a would therefore be the least common multiple of 3, 4, and 5, namely, 60. b must be the greatest common divisor of the 7, 14, and 21, namely, 7. If b were less than the GCD, the result would be larger, and if b were greater than the GCD, the result would not be an integer in all cases. Thus, $N=\boxed{\frac{60}{7}}$.

3. $M = \frac{10{,}000 \text{ miles}}{30 \text{ mpg}}\cdot \$2.70 = \$900$

$N = \frac{25 \text{ kwh}}{100 \text{ miles}}\cdot(\$0.14 \text{ per kwh})\cdot 9000+\frac{1000}{25}(\$2.70) = \$315 + \$108 = \$423$

$\rightarrow M-N = \boxed{477}$.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS - 2023 - SOLUTIONS

Round 2: Algebra 1

1. $\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{3}}<\sqrt{27}\rightarrow\frac{1}{\sqrt{x}}<\frac{1}{\sqrt{3}}+\sqrt{27}\rightarrow\frac{1}{\sqrt{x}}<\frac{1+\sqrt{81}}{\sqrt{3}}\rightarrow\frac{1}{\sqrt{x}}<\frac{10}{\sqrt{3}}$.

Then $\frac{1}{x}<\frac{100}{3}$ gives $\boxed{x>\frac{3}{100}}$.

Alternate Solution: Multiply both sides by $\sqrt{3}\cdot\sqrt{x}$.

That gives $\sqrt{3}-\sqrt{x}<\sqrt{81x}\rightarrow\sqrt{3}<10\sqrt{x}\rightarrow\sqrt{x}>\frac{\sqrt{3}}{10}\rightarrow\boxed{x>\frac{3}{100}}$.

2. Given: $\begin{cases}x+y-z &= -5\\ xy &= z\\ y-z=-9\end{cases}$

Substitute $y-z=-9$ into the first equation to obtain $x+(-9)=-5\rightarrow x=4$. The second equation becomes $4y=z$. Substitute into the 3rd equation, obtaining $y-4y=-9$, so $y=3$. The second equation then gives $4\cdot3=12=z$. Thus, $x+y+z=4+3+12=\boxed{19}$.

3. Their difference will have a solution in common so $\left(x^2+3x-k\right)-\left(x^2-5x-5k\right)=0\rightarrow$

$8x+4k=0\rightarrow k=-2x$. The first equation becomes $x^2+3x-(-2x)=0\rightarrow x(x+5)=0\rightarrow$ $x=0,-5$, confirmed by the second equation, which becomes

$x^2-5x-5(-2x)=0\rightarrow x(x+5)=0\rightarrow x=0,-5$.

Then $\begin{cases}k=-2x\\ x=0,-5\end{cases}\rightarrow k=\boxed{0,10}$.

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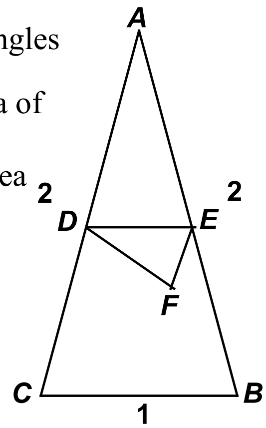
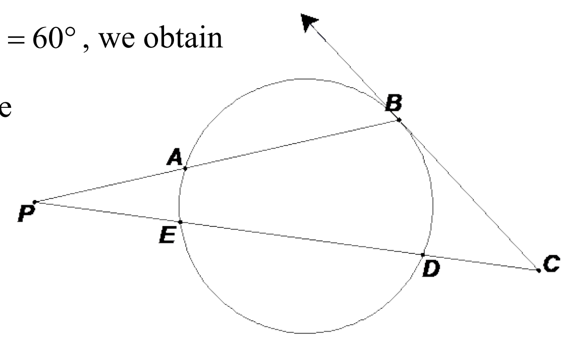
NEW ENGLAND PLAYOFFS - 2023 - SOLUTIONS

Round 3: Geometry

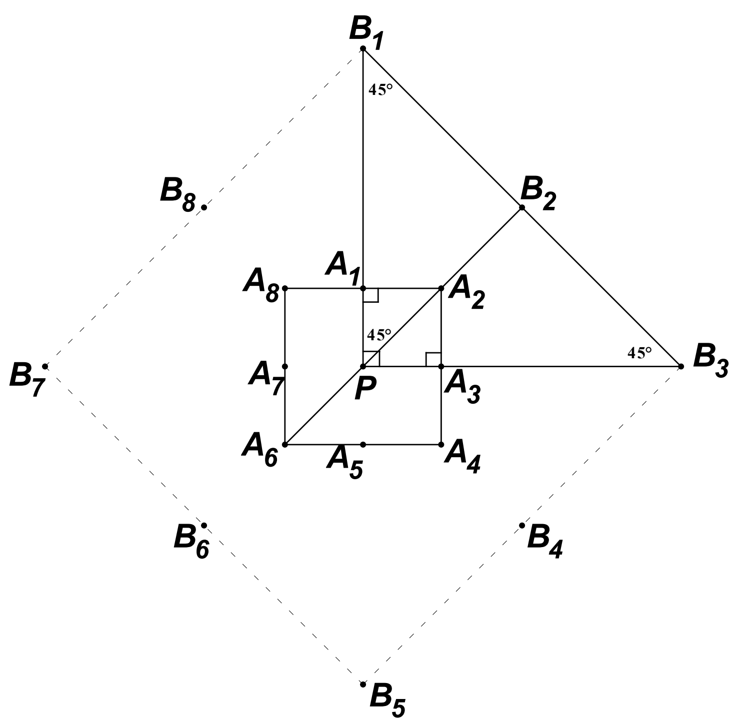
1. $\overline{DE}$ is a midline, so $DE = \frac{1}{2}$. Since both ΔABC and ΔDEF are isosceles triangles with $\angle A \cong \angle FDE$, $\Delta ABC \sim \Delta DEF \rightarrow$ the ratio of the area of ΔDEF to the area of ΔABC is $\left(\frac{1/2}{2}\right)^2 = \frac{1}{16}$. The height of ΔABC is $\sqrt{4-(1/2)^2} = \frac{\sqrt{15}}{2}$, so the area of ΔABC is $\frac{1}{2}\cdot 1\cdot\frac{\sqrt{15}}{2} = \frac{\sqrt{15}}{4}$. Thus, the area of ΔDEF is $\frac{\sqrt{15}}{4}\cdot\frac{1}{16} = \boxed{\frac{\sqrt{15}}{64}}$.



2. Let $m\overset{\frown}{AE} = x°$. Then $m\overset{\frown}{AB} = 4x°$, $m\overset{\frown}{AE} = 5x°$, and $m\overset{\frown}{BD} = 2x°$. Their sum is $12x°$, and from
$12x = 360°$, we obtain $x = 30°$. From $m\overset{\frown}{AE} = 30°$ and $m\overset{\frown}{BD} = 60°$, we obtain $m\angle P = \frac{60° - 30°}{2} = 15°$. Since $m\angle TBP = \frac{1}{2}\left(m\overset{\frown}{AB}\right)$, we have $m\angle TBP = 60°$. Then
$m\angle P + m\angle C = 60° \rightarrow 15° + m\angle C = 60° \rightarrow m\angle C = \boxed{45}$
.



3. $PA_1 = PA_3 = \frac{1}{2} \rightarrow PB_1 = PB_3 = 2$, making ΔPB_1B_3 an isosceles right triangle (i.e., a 45°-45°-90° triangle) with legs of length of 2, and hypotenuse $\overline{B_1B_3}$ of length $2\sqrt{2}$. Draw $\overrightarrow{PA_2}$ intersecting $\overline{B_1B_3}$ at B_2. Marked angles have the indicated angle measures $\rightarrow \angle PB_2B_1$ is a right angle; hence, $B_1B_2 = \sqrt{2}$. Using a similar argument, $B_2B_3 = B_3B_4 = ... = B_7B_8 = B_8B_1 = \sqrt{2}$ $\rightarrow$ the boundary of the region $B_1B_3B_5B_7$ is a square of side-length $2\sqrt{2}$. Thus, the area is $\boxed{8}$.



MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS - 2023 - SOLUTIONS

Round 4: Algebra 2

1. $\log_x 9 = -2 \rightarrow 9 = x^{-2} \rightarrow \frac{1}{x^2} = 9 \rightarrow x^2 = \frac{1}{9}$. Thus, $x = \boxed{\frac{1}{3}}$.

2. $g(x) = (3+x+1)(1-(2x+2)) - c = (x+4)(-2x-1) - c = -2x^2 - 9x - 4 - c$

Completing the square, we have

$$g(x) = -2\left(x^2 + \frac{9}{2}x + \frac{81}{16}\right) - 4 - c + \frac{81}{8} = -2\left(x + \frac{9}{4}\right)^2 + \frac{-32 - 8c + 81}{8} = -2\left(x + \frac{9}{4}\right)^2 + \frac{49 - 8c}{8} < 0.$$

Thus, the maximum is $\frac{49-8c}{8}$ (and it occurs for $x = -\frac{9}{4}$) $\rightarrow c_{\min} = \boxed{7}$.

3. There are four E's, three A's, two S's, and one each of B, H, I, L, N, P, R, T, and U; 18 letters in all. To find the total number of ways to arrange the 18 letters, we take the number of permutations of an 18-set (18!), then divide by the number of repeated arrangements, which are the number of ways to permute the letters E, A, and S in their respective groups (this will equal (4!)(3!)(2!)). Therefore, the total number of arrangements is $\frac{18!}{4!\cdot 3!\cdot 2!}$

Now we find the number of arrangements which have the phrase HASBEEN. To do this, we treat "HASBEEN" as a single character, with the other 11 characters being the 11 remaining letters (two E's, two A's, and one each of I, L, P, R, S, T, and U). The total number of ways to arrange these 12 characters is $\frac{12!}{2!\cdot 2!}$, taking into account the double-counts that happen with the A's and E's.

We now can solve for our probability, which will be the number of arrangements that have the phrase HASBEEN, divided by the total number of arrangements:

$$\frac{\frac{12!}{2!\cdot 2!}}{\frac{18!}{4!\cdot 3!\cdot 2!}} = \frac{12!\cdot 4!3!\cdot 2!}{18!\cdot 2!\cdot 2!} = \boxed{\frac{1}{185{,}640}}$$

Round 4: Algebra 2 - continued

3. **Alternate Solution:**

Consider all 18 letters to be distinguishable - $\text{ITHA}_1\text{S}_1\text{BE}_1\text{E}_2\text{NA}_2\text{PLE}_3\text{A}_3\text{S}_2\text{URE}_4$.

There are $18!$ different 18-letter permutations. There are 3 ways to select one A $(\text{A}_1, \text{A}_2, \text{A}_3)$.

There are 2 ways to select one S (S_1, S_2).

There are $4 \cdot 3 = 12$ ways to permute two E's from a group of four, namely,

$\text{E}_1\text{E}_2, \text{E}_1\text{E}_3, \text{E}_1\text{E}_4, \text{E}_2\text{E}_1, \text{E}_2\text{E}_3, \text{E}_2\text{E}_4, ..., \text{E}_4\text{E}_3$.

There are 7 letters in the string HASBEEN, so there are 11 distinguishable letters left over, and those can be arranged in $11!$ ways. Finally, the string HASBEEN can be placed before these 11 letters, after the 11 letters, or in any one of the 10 places between them, for a total of 12 positions.

For example,

HASBEEN _ _ _ _ _ _ _ _ _ _ _

_ _ _ _ _ _ _ _ _ _ _ HASBEEN

_ HASBEEN _ _ _ _ _ _ _ _ _ _

. . .

_ _ _ _ _ _ _ _ _ _ HASBEEN _

The probability that HASBEEN occurs is, therefore,

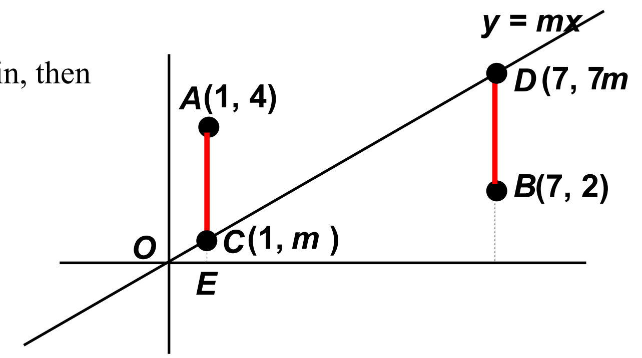
$$\frac{\left(\overset{H}{1}\cdot\overset{A}{3}\cdot\overset{S}{2}\cdot\overset{B}{1}\cdot\overset{EE}{12}\cdot\overset{N}{1}\cdot 11!\right)\cdot 12}{18!} = \frac{3\cdot 2\cdot \cancel{12}^{3}}{18\cdot 17\cdot \cancel{16}^{4}\cdot 15\cdot 14\cdot 13} = \frac{1}{17\cdot 4\cdot 15\cdot 14\cdot 13} = \boxed{\mathbf{\frac{1}{185,640}}}.$$

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS - 2023 - SOLUTIONS

Round 5: Analytic Geometry

1. If m is the slope of the line through the origin, then $7m-2=4-m \rightarrow 8m=6 \rightarrow m=\boxed{\frac{3}{4}}$.



Alternate Solution
Let C be on the line a distance of t below A, and let D be on the line a distance of t above B.
Then $C=(1,4-t)$ and $D=(7,2+t)$. The slope of $\overline{CD}=m=\frac{(2+t)-(4-t)}{6}=\frac{t-1}{3}$.
But, using ΔOCE, $m=\frac{4-t}{1}$. Equating, $4-t=\frac{t-1}{3} \rightarrow t=\frac{13}{4} \rightarrow m=4-\frac{13}{4}=\boxed{\frac{3}{4}}$.

2. f and f^{-1} intersect on the line $y=x$; thus, the point of intersection is $(-5,-5)$. Substituting gives $-5=m(-5)+k \rightarrow m=\frac{k+5}{5}$. Then $\frac{k}{5}+1<100 \rightarrow k<495$. Since m is an integer, k must be divisible by 5; so, $k=\boxed{490}$.

3. Let $P=(a,-(a+1)(a-5))$, $A=(-1,0)$, and $B=(5,0)$.
The slope of $AP=\frac{-(a+1)(a-5)-0}{a-(-1)}=5-a$. The slope of $PB=\frac{-(a+1)(a-5)-0}{a-5}=-(a+1)$.
Then $-(a+1)(5-a)=-1 \rightarrow a^2-4a-4=0$.
Thus, $a=\frac{4\pm\sqrt{4^2-4(1)(-4)}}{2}=\frac{4\pm\sqrt{32}}{2}=\boxed{2\pm2\sqrt{2}}$.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS - 2023 - SOLUTIONS

Round 6: Trig and Complex Numbers

1. Note that $32 \cdot 11 = 352$, so the angle whose measure is 11θ lies in the 4th quadrant. Since the cosines are equal, $\theta + 11\theta = 360° \rightarrow 12\theta = 360° \rightarrow \theta = \boxed{30}$.

2. $\cos(2x) - \tan(x) = 1 \rightarrow 1 - 2\sin^2 x - \tan x = 1 \rightarrow 2\sin^2 x + \dfrac{\sin x}{\cos x} = 0 \rightarrow \sin x(2\sin x \cos x + 1) = 0$ for $\cos x \neq 0$, i.e., for $x \neq \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$. Then either $\sin x = 0$ or $\sin(2x) = -1$.

   The former yields $x = 0$ or π; the latter yields $2x = \dfrac{3\pi}{2} + 2\pi k \rightarrow x = \dfrac{3\pi}{4} + \pi k \rightarrow$ $x = \boxed{0, \dfrac{3\pi}{4}, \pi, \dfrac{7\pi}{4}}$.

3. $A^3 = \dfrac{1}{8}\left(\cos 150° + i\sin 150°\right)$ while $B^5 = 32\left(\cos 90° + i\sin 90°\right)$. Thus, we have a triangle with adjacent sides of $\dfrac{1}{8}$ and 32 enclosing an angle of 150° – 90° = 60°.

   Its area is $\dfrac{1}{2} \cdot \dfrac{1}{8} \cdot 32\sin 60 = 2 \cdot \dfrac{\sqrt{3}}{2} = \boxed{\sqrt{3}}$.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS - 2023 - SOLUTIONS

Team Round

1. i) If M_0 has 3 odd digits and 1 even digit as in OOOE, $M_1 = 3123$
$\rightarrow M_2 = 3123 \rightarrow M_{2023} = 3123$.

ii) If M_0 has 3 even digits and 1 odd digit as in EEEO, $M_1 = 1323$
$\rightarrow M_2 = 3123 \rightarrow M_{2023} = 3123$.

iii) If M_0 has 2 odd and 2 even digits as in OOEE, $M_1 = 2204$, a number with 4 even digits
$\rightarrow M_2 = 0440 \rightarrow M_{2023} = 0440 = 440$.

iv) If M_0 has 4 even digits, $M_{2023} = 440$.

v) If M_0 has 4 odd digits, then $M_1 = 4040$, a number with 4 even digits $\rightarrow M_{2023} = 440$.

Thus, the required sum is 3123 + 440 = **3563**.

Team Round - continued

2. D_1 contains one 1, two 2s, three 3s, etc.

This listing contains $1+2+3+...+25=\frac{25\cdot 26}{2}$ integers.

The sum of the terms is $1+2^2+3^2+4^2+...+25^2$.

Using the summation formula $1+2^2+3^2+...+n^2=\frac{n(n+1)(2n+1)}{6}$, we have $\frac{25\cdot 26\cdot 51}{6}$.

Without calculating either of these quotients, $m_1=\dfrac{\frac{\cancel{25}\cdot\cancel{26}\cdot 51}{\cancel{6}^{3}}}{\frac{\cancel{25}\cdot\cancel{26}}{\cancel{2}}}=17$.

D_2 contains one 1, two 2s, four 3s, eight 4s, etc. 25 occurs 2^{24} times.

Before the first 25, there are $2^0+2^1+2^3+2^4+...+2^{23}=1\left(\frac{2^{24}-1}{2-1}\right)=2^{24}-1$ integers.

Thus, the number of 25s is one more than the number of smaller numbers in this listing.
So , if the numbers are arranged in nondecreasing order, we have

2^{24} - 1 integers		2^{24} - 1 integers	
1 2 2 3 3 3 3 ... 24	25	25s	$\Rightarrow m_2 = 25$.

D_3 : Since n and $\phi(n)$ are not independent, we are looking for a large value of n that has a relatively small $\phi(n)$ value. Suppose the prime factorization of n is $p_1^{e_1}\cdot p_2^{e_2}\cdot...\cdot p_k^{e_k}$.

$\phi(n)$ can be calculated by simply making a list or by invoking the following formula:

$\phi(n)=n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdot...\cdot\left(1-\frac{1}{p_k}\right)$. Clearly, $\phi(25)$ is relatively large, since most integers between 1 and 25, inclusive, are not divisible by 5; in fact, $\phi(25)=20\rightarrow 25$ occurs only 5 times.

For $n=24$, only $1,5,7,11,13,17,19,23$ are relatively prime to $24\Rightarrow\phi(24)=8$; alternatively,

$24=2^3\cdot 3^1\Rightarrow\phi(24)=24\cdot\frac{1}{2}\cdot\frac{2}{3}=8$. $24-\phi(24)=16\rightarrow m_3=24\rightarrow(m_1,m_2,m_3)=\boxed{\mathbf{(17,25,24)}}$.

The following chart confirms that 16 is the largest frequency and that the mode is 24.

n	$\phi(n)$	Freq	n	$\phi(n)$	Freq	N	$\phi(n)$	Freq	N	$\phi(n)$	Freq	N	$\phi(n)$	Freq
1	1	0	6	2	4	11	10	1	16	8	8	21	12	9
2	1	1	7	6	1	12	4	8	17	16	1	22	10	12
3	2	1	8	4	4	13	12	1	18	6	12	23	22	1
4	2	2	9	6	3	14	6	8	19	18	1	24	8	16
5	4	1	10	4	6	15	8	7	20	8	12	25	20	5

Team Round - continued

3. Let ℓ be the slant height. From $\pi r\ell = \frac{1}{3}\pi r^2 h$ we obtain $3\ell = rh \to 3\sqrt{r^2+h^2} = rh$. Squaring gives $9r^2 + 9h^2 = r^2h^2 \to r^2 = \frac{9h^2}{h^2-9}$. To find integer values of h^2 that make r an integer, let $t = h^2 - 9$, making $9h^2 = 9(t+9)$. Then $r^2 = \frac{9(t+9)}{t} = 9 + \frac{81}{t}$. Clearly, t must be a factor of 81. If $t = 1$, then $r^2 = 90$, but if $t = 3$, then $r^2 = 36 \to r = 6$ and that will give the least integer value of h^2 that makes r an integer. Thus, $3 = h^2 - 9$, so $h^2 = \boxed{12}$.

4. Under the given conditions (4 mathletes in each round, a mathlete from each class in each round, no mathlete competing in more than 2 rounds), how many rosters like the following are possible?

		Round 1	Round 2	Round 3
freshmen	A			X
	B	X	X	
sophomores	C		X	
	D	X		X
juniors	E		X	X
	F	X		X
	G	X	X	
seniors	H		X	X
	I	X		X
	J	X	X	

⇒

Round 1: ACEH
Round 2: ADFI
Round 3: BCGJ

Analysis #1
Over the three rounds the freshmen will have permutations of AAB or BBA, a total of 6 possibilities. The sophomores will also have 6 possible choices for the group. The juniors could have permutations of EEF, FFE, EEG, GGE, FFG, and GGF. Each of these groups can be arranged in 3 ways, making for a total of 18 different choices. But the juniors could also have permutations of EFG, adding another 6 choices for a total of 24. The same is true of the seniors. So the coach can enter $6 \cdot 6 \cdot 24 \cdot 24 =$ $\boxed{\mathbf{20,736}}$ possible rosters.

Team Round - continued

4. ***Analysis #2***

Start with the possibilities for placing the two freshmen. There are three rounds, and neither student can participate all three times. Without the last restriction, there would be $2^3 = 8$ possibilities for placing the freshman (choose one of the two students for each round), but there are two possibilities that must now be ruled out (those which place the same freshman in all three rounds), so there are 6 total ways to place the freshman. Likewise, there are 6 ways to place the sophomores.

For the case of the juniors, there are three students. Without the restriction on how many rounds an individual can be in, there would be $3^3 = 27$ different ways to place the juniors (choose one of three students for each round), but there are now three possibilities which must be ruled out (those which place the same junior in each round) so there are 24 ways to place the juniors. Likewise, there are 24 ways to place the seniors.

Since the freshmen roster decisions, for example, are independent of the junior roster decisions, for example, there are 6 x 6 x 24 x 24 = $\boxed{\mathbf{20,736}}$ possible rosters.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

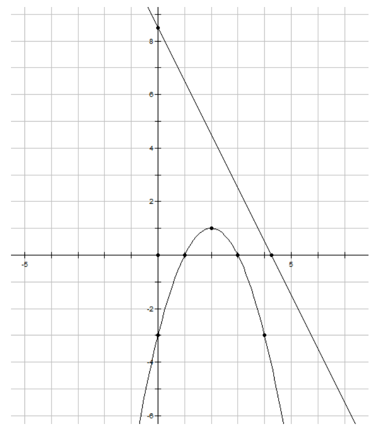
NEW ENGLAND PLAYOFFS - 2023 - SOLUTIONS

Team Round - continued

5. Let $A\left(a,-a^2+4a-3\right)$ be a point on the parabola. Then the distance to the line is given by

$$\frac{\left|4a+2\left(-a^2+4a-3\right)-17\right|}{\sqrt{4^2+2^2}}.$$ This simplifies to $$\frac{\left|-2a^2+12a-23\right|}{2\sqrt{5}}=\frac{\left|-2\left(a^2-6a\right)-23\right|}{2\sqrt{5}}=$$

$$\frac{\left|-2\left(a^2-6a+9\right)-23+18\right|}{2\sqrt{5}}=\frac{\left|-2\left(a-3\right)^2-5\right|}{2\sqrt{5}}.$$ If $a=3$, the expression takes on its minimum value of $\frac{5}{2\sqrt{5}}=\boxed{\frac{\sqrt{5}}{2}}$.



Alternate solution: Find the equation of the line parallel to the given line that is tangent to the parabola. Then find the distance from that line to the given line:

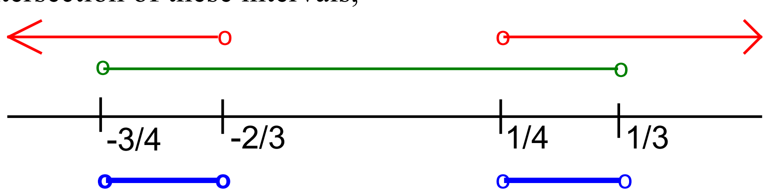
Based on $y=-2x+\frac{17}{2}$, we use the equation $y=-2x+k$. From $-x^2+4x-3=-2x+k$ we obtain $0=x^2-6x+\left(k+3\right)$. This takes on its minimum value if $k+3=9\rightarrow k=6$. From the graph of $y=-2x+6$, we choose the point (0, 6) and find its distance to $4x+2y-17=0$.

6. $0<2\theta<\frac{\pi}{2}\rightarrow,\ 0<\theta<\frac{\pi}{4}\rightarrow 0<\tan\theta<1$

$\rightarrow(4x-1)(3x+2)>0\rightarrow x>\frac{1}{4}$ or $x<-\frac{2}{3}$;

$(4x-1)(3x+2)<1\rightarrow 12x^2+5x-3<0\rightarrow(4x+3)(3x-1)<0\rightarrow-\frac{3}{4}<x<\frac{1}{3}$, and

Taking the intersection of these intervals,



we have the x-values that satisfy both conditions, namely, $\boxed{-\frac{3}{4}<x<-\frac{2}{3}\ \text{or}\ \frac{1}{4}<x<\frac{1}{3}}$.