CT ARML Team, 2023

Team Selection Test 1

1. Let the *n*th Euclid-ish number be the number that is one less than the product of the first *n* prime numbers. What is the smallest composite Euclid-ish number?
[Answer: 209]

2. A hand of four cards of the form $(a, a, a+1, a+1)$ is called a "straight". Dominic has a deck consisting of four colors of each of the cards numbered 2, 3, 4, and 5. If the deck is shuffled, and Dominic draws four cards from the deck, the probability that they form a "straight" is $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.
[Answer: 482]

3. Let a and b be real numbers such that the polynomial

$$P(x) = x^3 - ax^2 + 9ax - b$$

has three distinct real roots in geometric progression. Compute b.
[Answer: 729]

4. Compute

$$\frac{2^{\log_{2/3} 2}}{3^{\log_{2/3} 3}}$$

[Answer: 6]

5. Let *ABCD* be a square of side length 2. Equilateral triangles *ABP*, *BCQ*, *CDR*, and *DAS* are constructed inside the square. The area of quadrilateral *PQRS* can be expressed as $a - b\sqrt{c}$, where $a, b,$ and c are positive integers, and c is not divisible by the square of any prime number. Compute $a+b+c$.
[Answer: 15]

6. Suppose that $x \in (0, 2\pi)$ satisfies $\sin(x) + \sin(2x) = \sin(4x)$. The largest possible value of x can be expressed in the form $\frac{m\pi}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.
[Answer: 26]

7. What is the smallest positive integer k such that $2023k + 1$ is a perfect square?
[Answer: 41]

8. Let $\{a_n\}$ be a sequence defined recursively by the equations $a_1 = 2$ and $a_n = 2a_{n-1} + n$. The sum below can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

$$\sum_{n=1}^{\infty} \frac{a_n}{3^n}$$

[Answer: 17]

9. Find the number of positive integers n, with $n > 2023$, such that

$$m = \frac{1^2 + 2^2 + 3^2 + \cdots + n^2}{1 + 2 + 3 + \cdots + n}$$

is an integer with $m < 2023$.
[Answer: 336]

10. At the start of a game, there are ten girls standing equally spaced on the perimeter of a circle. Each girl randomly chooses someone (possibly herself) to "follow" for the entire game. Every second, each girl moves to the position previously occupied by the girl she is following. That is, if girl a is at position x at time t seconds, anyone following a will be at position x at time $t + 1$ seconds. (Consequently, there can be multiple girls at any given position at any given time, apart from at the start of the game.) The probability that all the girls are in their original positions after 120 seconds is $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute the sum of the number of positive factors of m and the number of positive factors of n.
[Answer: 33]

CT ARML Team, 2023

Team Selection Test 1 Solution

1. Let the *n*th Euclid-ish number be the number that is one less than the product of the first *n* prime numbers. What is the smallest composite Euclid-ish number?

Solution:

$$2 \times 3 - 1 = 5$$

$$2 \times 3 \times 5 - 1 = 29$$

$$2 \times 3 \times 5 \times 7 - 1 = 209 = 11 \times 19$$

So the answer is $\boxed{209}$.

2. A hand of four cards of the form $(a, a, a+1, a+1)$ is called a "straight". Dominic has a deck consisting of four colors of each of the cards numbered 2, 3, 4, and 5. If the deck is shuffled, and Dominic draws four cards from the deck, the probability that they form a "straight" is $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.

Solution:
It is easy to see that the values of the cards of a "straight" must be (2, 2, 3, 3), or (3, 3, 4, 4), or (4, 4, 5, 5).
Let's look at one case (2, 2, 3, 3). We have $\binom{4}{2}$ ways to select two 2s and $\binom{4}{2}$ ways to select two 3s, and 4! ways to order them. So we have a total of $3 \times 4! \times \binom{4}{2} \times \binom{4}{2}$ ways to draw a "Straight". There are total of $16 \times 15 \times 14 \times 13$ ways to draw 4 cards.

$$P(Straight) = \frac{3 \times 24 \times 6 \times 6}{16 \times 15 \times 14 \times 13} = \frac{27}{455}$$

So the answer is 27+455 = $\boxed{482}$.

3. Let a and b be real numbers such that the polynomial

$$P(x) = x^3 - ax^2 + 9ax - b$$

has three distinct real roots in geometric progression. Compute *b*.

Solution:
Let the three roots of the expression be $\frac{r}{k}, r, kr$. By Vieta,

$$\frac{r}{k} + r + kr = r\left(\frac{1}{k} + 1 + k\right) = a$$

$$\frac{r}{k} \cdot r + r \cdot kr + kr \cdot \frac{r}{k} = r^2\left(\frac{1}{k} + 1 + k\right) = 9a$$

Solve to obtain $r = 9$.
Use Vieta one more time we have $b = \frac{r}{k} \cdot r \cdot kr = r^3 = \boxed{729}$

4. Compute

$$\frac{2^{\log_{2/3} 2}}{3^{\log_{2/3} 3}}$$

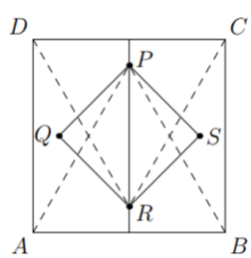
Solution:
Let $m = \log_{2/3} 2$ and $n = \log_{2/3} 3$. We can see $m - n = \log_{\frac{2}{3}} 2 - \log_{\frac{2}{3}} 3 = 1$.
From $m = \log_{2/3} 2$, we can have $2 = (\frac{2}{3})^m$ and $2^m = 2 \cdot 3^m$

$$\frac{2^{\log_{2/3} 2}}{3^{\log_{2/3} 3}} = \frac{2^m}{3^n} = \frac{2 \cdot 3^m}{3^n} = 2 \cdot 3^{(m-n)} = 6$$

The answer is $\boxed{6}$.

5. Let *ABCD* be a square of side length 2. Equilateral triangles *ABP*, *BCQ*, *CDR*, and *DAS* are constructed inside the square. The area of quadrilateral *PQRS* can be expressed as $a - b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime number. Compute $a + b + c$.

Solution:



Triangle PAB is an equilateral triangle so the distance from P to AB is $2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$. The distance from R to CD is also $\sqrt{3}$.

$$PR = QS = 2 - 2 \cdot \left(2 - \sqrt{3}\right) = 2\sqrt{3} - 2$$

PQRS is a rhombus. So,

$$S_{PQRS} = \frac{1}{2}(2\sqrt{3} - 2)^2 = 8 - 4\sqrt{3}$$

So the answer is $\boxed{15}$.

6. Suppose that $x \in (0, 2\pi)$ satisfies $\sin(x) + \sin(2x) = \sin(4x)$. The largest possible value of x can be expressed in the form $\frac{m\pi}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

<u>Solution:</u>

$$\sin x = \sin(4x) - \sin(2x) = \sin(3x + x) - \sin(3x - x)$$
$$= \sin(3x)\cos x + \cos(3x)\sin x - (\sin(3x)\cos x - \cos(3x)\sin x) = 2\cos(3x)\sin x$$

Case 1: $\sin x = 0, and\ x = \pi$

Case 2: $\sin x \neq 0, we\ can\ divide\ by\ \sin x\ on\ each\ side\ to\ obtain\ \cos(3x) = \frac{1}{2}$. We can get

$$3x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3}$$

The maximum solution is $x = \frac{17\pi}{9}$. The answer is $\boxed{26}$.

7. What is the smallest positive integer k such that $2023k + 1$ is a perfect square?

<u>Solution:</u>

$$2023k + 1 = n^2$$
$$2023k = n^2 - 1 = (n + 1)(n - 1)$$

Since 2023 = 7×17^2, we can try a few combinations.

Case 1: $7 \times 17 = n + 1\ or\ n - 1, and\ 17k = n - 1\ or\ n + 1$. We cannot get any integer solution.

Case 2: $17^2 = n + 1\ or\ n - 1, and\ 7k = n - 1\ or\ n + 1$. We can find one solution with $n = 288\ and\ k = 41$.

The answer is $\boxed{41}$.

8. Let $\{a_n\}$ be a sequence defined recursively by the equations $a_1 = 2$ and $a_n = 2a_{n-1} + n$. The sum below can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

$$\sum_{n=1}^{\infty} \frac{a_n}{3^n}$$

<u>Solution:</u>

Because $a_n = 2a_{n-1} + n$, we can add n to each side and obtain

$$a_n + n = 2a_{n-1} + 2n = 2\big(a_{n-1} + (n - 1)\big) + 2$$

Let $a_n = b_n - n$, we can convert the above expression as

$$b_n = 2b_{n-1} + 2$$

Similarly we add 2 at each side and let $c_n = b_n + 2$, we have

$$c_n = b_n + 2 = 2(b_{n-1} + 2) = 2c_{n-1}$$

As $c_n = 2c_{n-1}$ and $c_1 = b_1 + 2 = a_1 + 1 + 2 = 5, we\ can\ easily\ get$
$c_n = 5 \cdot 2^{(n-1)}$. Thus $a_n = 5 \cdot 2^{(n-1)} - 2 - n$

$$\sum_{n=1}^{\infty} \frac{a_n}{3^n} = \sum_{n=1}^{\infty} \frac{5 \cdot 2^{(n-1)} - 2 - n}{3^n}$$
$$= \frac{5}{2}\sum_{n=1}^{\infty} (\frac{2}{3})^n - 2\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n - \sum_{n=1}^{\infty} \frac{n}{3^n}$$

The 1st and 2nd term are infinite geometric series and the sums are 5, and 1 respectively. The 3rd term is a little more complicated but can be done quickly as follows.

$$S = \frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \frac{4}{81} + \cdots$$
$$\frac{1}{3}S = \quad \frac{1}{9} + \frac{2}{27} + \frac{3}{81} + \cdots$$

Subtract we get

$$\frac{2}{3}S = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \frac{1}{2}$$
$$S = \frac{3}{4}$$

The sum is 5 – 1 – ¾ = 13/4. So the answer is $\boxed{17}$.

9. Find the number of positive integers n, with $n > 2023$, such that

$$m = \frac{1^2 + 2^2 + 3^2 + \cdots + n^2}{1 + 2 + 3 + \cdots + n}$$

is an integer with $m < 2023$.

Solution:

$$m = \frac{1^2 + 2^2 + 3^2 + \cdots + n^2}{1 + 2 + 3 + \cdots + n} = \frac{\frac{n(n+1)(2n+1)}{6}}{\frac{n(n+1)}{2}} = \frac{2n+1}{3}$$

So $2n + 1 = 3m < 3 \cdot 2023 = 6069$. Hence $n < 3034$.
On the other hand, m is an integer, so $2n \equiv -1 (mod\ 3) = 2 (mod\ 3), and\ n \equiv 1 (mod\ 3)$.
Since $2023 < n < 3034, n = 2026, 2029, \dots, 3031,$ the number of positive integer n is $\boxed{336}$.

10. At the start of a game, there are ten girls standing equally spaced on the perimeter of a circle. Each girl randomly chooses someone (possibly herself) to "follow" for the entire game. Every second, each girl moves to the position previously occupied by the girl she is following. That is, if girl a is at position x at time t seconds, anyone following a will be at position x at time $t + 1$ seconds. (Consequently, there can be multiple girls at any given position at any given time, apart from at the start of the game.) The probability that all the girls are in their original positions after 120 seconds is $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute the sum of the number of positive factors of m and the number of positive factors of n.

<u>Solution:</u>

- Total possible ways are 10^{10} because each girl can choose any the 10 positions to follow (including themselves.)
- Now we need to find the desired outcome that every girl ends up in their original spot after 120 seconds. First, we can tell that anytime when two girls follow the same person they will never be able to go back to their original spots simultaneously. This means we just need to consider permutations only.
- Among the 10! possible permutations, we will need to remove the permutation cycle of length 7 or 9 because they are not factors of 120 and each girl will not return to their original position after 120 seconds.
- The number of possible 7-cycles is $\binom{10}{7} \times 3! \times 6!$; the first two terms represent the process of choosing the 7 people in the 7-cycle and permuting the 3 people not in the 7-cylce, and the last term represents the number of ways to create a cycle of 7 people.
- Similarly the number of possible 9-cycle is $\binom{10}{9} \times 1! \times 8!$.
- The total number of desired outcome is

$$10! - \binom{10}{7} \times 3! \times 6! - \binom{10}{9} \times 1! \times 8! = 10! \times \left(1 - \frac{1}{7} - \frac{1}{9}\right) = 10! \times \frac{47}{63}$$

$$P = \frac{10! \times \frac{47}{63}}{10^{10}} = \frac{3^2 \times 47}{2^2 \times 5^8}$$

- So *m* has 6 factors and *n* has 27 factors and the answer is $\boxed{33}$.

1 Team Problems

Problem 1. There exists a digit Y such that, for any digit X, the seven-digit number $\underline{1}\,\underline{2}\,\underline{3}\,\underline{X}\,\underline{5}\,\underline{Y}\,\underline{7}$ is not a multiple of 11. Compute Y.

Problem 2. A point is selected at random from the interior of a right triangle with legs of length $2\sqrt{3}$ and 4. Let p be the probability that the distance between the point and the nearest vertex is less than 2. Then p can be written in the form $a + \sqrt{b}\pi$, where a and b are rational numbers. Compute (a, b).

Problem 3. The square $ARML$ is contained in the xy-plane with $A = (0, 0)$ and $M = (1, 1)$. Compute the length of the shortest path from the point $(2/7, 3/7)$ to itself that touches three of the four sides of square $ARML$.

Problem 4. For each positive integer k, let S_k denote the infinite arithmetic sequence of integers with first term k and common difference k^2. For example, S_3 is the sequence $3, 12, 21, \ldots$. Compute the sum of all k such that 306 is an element of S_k.

Problem 5. Compute the sum of all values of k for which there exist positive real numbers x and y satisfying the following system of equations.

$$\left\{\begin{array}{rcl} \log_x y^2 + \log_y x^5 & = & 2k - 1 \\ \log_{x^2} y^5 - \log_{y^2} x^3 & = & k - 3 \end{array}\right.$$

Problem 6. Let $W = (0, 0)$, $A = (7, 0)$, $S = (7, 1)$, and $H = (0, 1)$. Compute the number of ways to tile rectangle $WASH$ with triangles of area 1/2 and vertices at lattice points on the boundary of $WASH$.

Problem 7. Compute $\sin^2 4^\circ + \sin^2 8^\circ + \sin^2 12^\circ + \cdots + \sin^2 176^\circ$.

Problem 8. Compute the area of the region defined by $x^2 + y^2 \leq |x| + |y|$.

Problem 9. The arithmetic sequences $a_1, a_2, a_3, \ldots, a_{20}$ and $b_1, b_2, b_3, \ldots, b_{20}$ consist of 40 distinct positive integers, and $a_{20} + b_{14} = 1000$. Compute the least possible value for $b_{20} + a_{14}$.

Problem 10. Compute the ordered triple (x, y, z) representing the farthest lattice point from the origin that satisfies $xy - z^2 = y^2 z - x = 14$.

2 Answers to Team Problems

Answer 1. 4

Answer 2. $\left(\frac{1}{4}, \frac{1}{27}\right)$

Answer 3. $\frac{2}{7}\sqrt{53}$

Answer 4. 326

Answer 5. $\frac{43}{48}$

Answer 6. 3432

Answer 7. $\frac{45}{2}$

Answer 8. $2 + \pi$

Answer 9. 10

Answer 10. $(-266, -3, -28)$

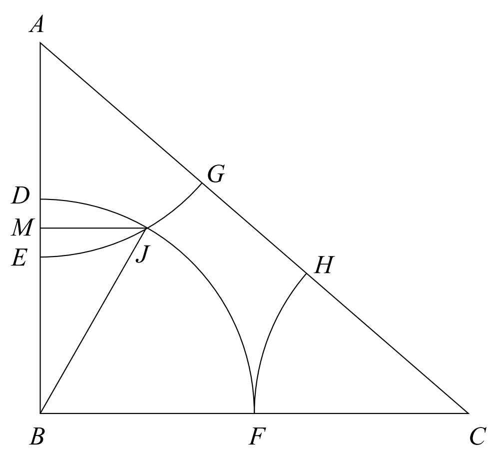
3 Solutions to Team Problems

Problem 1. There exists a digit Y such that, for any digit X, the seven-digit number $\underline{1}\,\underline{2}\,\underline{3}\,\underline{X}\,\underline{5}\,\underline{Y}\,\underline{7}$ is not a multiple of 11. Compute Y.

Solution 1. Consider the ordered pairs of digits (X, Y) for which $\underline{1}\,\underline{2}\,\underline{3}\,\underline{X}\,\underline{5}\,\underline{Y}\,\underline{7}$ **is** a multiple of 11. Recall that a number is a multiple of 11 if and only if the alternating sum of the digits is a multiple of 11. Because $1 + 3 + 5 + 7 = 16$, the sum of the remaining digits, namely $2 + X + Y$, must equal 5 or 16. Thus $X + Y$ must be either 3 or 14, making $X = 3 - Y$ (if $Y = 0$, 1, 2, or 3) or $14 - Y$ (if $Y = 5$, 6, 7, 8, or 9). Thus a solution (X, Y) exists unless $Y = \mathbf{4}$.

Problem 2. A point is selected at random from the interior of a right triangle with legs of length $2\sqrt{3}$ and 4. Let p be the probability that the distance between the point and the nearest vertex is less than 2. Then p can be written in the form $a + \sqrt{b}\pi$, where a and b are rational numbers. Compute (a, b).

Solution 2. Label the triangle as $\triangle ABC$, with $AB = 2\sqrt{3}$ and $BC = 4$. Let D and E lie on $\overline{AB}$ such that $DB = AE = 2$. Let F be the midpoint of $\overline{BC}$, so that $BF = FC = 2$. Let G and H lie on $\overline{AC}$, with $AG = HC = 2$. Now draw the arcs of radius 2 between E and G, D and F, and F and H. Let the intersection of arc DF and arc EG be J. Finally, let M be the midpoint of $\overline{AB}$. The completed diagram is shown below.



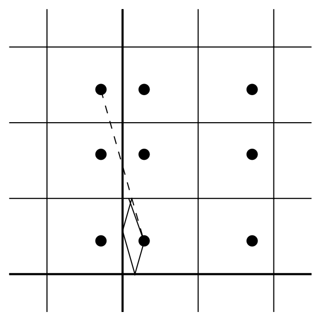
The region R consisting of all points within $\triangle ABC$ that lie within 2 units of any vertex is the union of the three sectors EAG, DBF, and FCH. The angles of these sectors, being the angles $\angle A, \angle B$, and $\angle C$, sum to $180°$, so the sum of their areas is 2π. Computing the area of R requires subtracting the areas of all intersections of the three sectors that make up R.

The only sectors that intersect are EAG and DBF. Half this area of intersection, the part above $\overline{MJ}$, equals the difference between the areas of sector DBJ and of $\triangle MBJ$. Triangle MBJ is a $1 : \sqrt{3} : 2$ right triangle because $BM = \sqrt{3}$ and $BJ = 2$, so the area of $\triangle MBJ$ is $\frac{\sqrt{3}}{2}$. Sector DBJ has area $\frac{1}{12}(4\pi) = \frac{\pi}{3}$, because $m\angle DBJ = 30°$. Therefore the area of intersection of the sectors is $2(\frac{\pi}{3} - \frac{\sqrt{3}}{2}) = \frac{2\pi}{3} - \sqrt{3}$. Hence the total area of R is $2\pi - (\frac{2\pi}{3} - \sqrt{3}) = \frac{4\pi}{3} + \sqrt{3}$.

The total area of $\triangle ABC$ is $4\sqrt{3}$, therefore the desired probability is $\frac{\frac{4\pi}{3}+\sqrt{3}}{4\sqrt{3}} = \frac{\pi}{3\sqrt{3}} + \frac{1}{4}$. Then $a = \frac{1}{4}$ and $b = \left(\frac{1}{3\sqrt{3}}\right)^2 = \frac{1}{27}$, hence the answer is $\left(\mathbf{\frac{1}{4}}, \mathbf{\frac{1}{27}}\right)$.

Problem 3. The square $ARML$ is contained in the xy-plane with $A = (0,0)$ and $M = (1,1)$. Compute the length of the shortest path from the point $(2/7, 3/7)$ to itself that touches three of the four sides of square $ARML$.

Solution 3. Consider repeatedly reflecting square $ARML$ over its sides so that the entire plane is covered by copies of $ARML$. A path starting at $(2/7, 3/7)$ that touches one or more sides and returns to $(2/7, 3/7)$ corresponds to a straight line starting at $(2/7, 3/7)$ and ending at the image of $(2/7, 3/7)$ in one of the copies of $ARML$. To touch three sides, the path must cross three lines, at least one of which must be vertical and at least one of which must be horizontal.



If the path crosses two horizontal lines and the line $x = 0$, it will have traveled a distance of 2 units vertically and 4/7 units vertically for a total distance of $\sqrt{2^2 + (4/7)^2}$ units. Similarly, the total distance traveled when crossing two horizontal lines and $x = 1$ is $\sqrt{2^2 + (10/7)^2}$, the total distance traveled when crossing two vertical lines and $y = 0$ is $\sqrt{2^2 + (6/7)^2}$, and the total distance traveled when crossing two vertical lines and $y = 1$ is $\sqrt{2^2 + (8/7)^2}$. The least of these is

$$\sqrt{2^2 + (4/7)^2} = \mathbf{\frac{2}{7}\sqrt{53}}.$$

Problem 4. For each positive integer k, let S_k denote the infinite arithmetic sequence of integers with first term k and common difference k^2. For example, S_3 is the sequence $3, 12, 21, \ldots$. Compute the sum of all k such that 306 is an element of S_k.

Solution 4. If 306 is an element of S_k, then there exists an integer $m \geq 0$ such that $306 = k + mk^2$. Thus $k \mid 306$ and $k^2 \mid 306 - k$. The second relation can be rewritten as $k \mid 306/k - 1$, which implies that $k \leq \sqrt{306}$ unless $k = 306$. The prime factorization of 306 is $2 \cdot 3^2 \cdot 17$, so the set of factors of

306 less than $\sqrt{306}$ is $\{1, 2, 3, 6, 9, 17\}$. Check each in turn:

$$\begin{aligned} 306 - 1 = 305, &\qquad 1^2 \mid 305 \\ 306 - 2 = 304, &\qquad 2^2 \mid 304 \\ 306 - 3 = 303, &\qquad 3^2 \nmid 303 \\ 306 - 6 = 300, &\qquad 6^2 \nmid 300 \\ 306 - 9 = 297, &\qquad 9^2 \nmid 297 \\ 306 - 17 = 289, &\qquad 17^2 \mid 289. \end{aligned}$$

Thus the set of possible k is $\{1, 2, 17, 306\}$, and the sum is $1 + 2 + 17 + 306 = \mathbf{326}$.

Problem 5. Compute the sum of all values of k for which there exist positive real numbers x and y satisfying the following system of equations.

$$\left\{ \begin{array}{rcl} \log_x y^2 + \log_y x^5 & = & 2k - 1 \\ \log_{x^2} y^5 - \log_{y^2} x^3 & = & k - 3 \end{array} \right.$$

Solution 5. Let $\log_x y = a$. Then the first equation is equivalent to $2a + \frac{5}{a} = 2k - 1$, and the second equation is equivalent to $\frac{5a}{2} - \frac{3}{2a} = k - 3$. Solving this system by eliminating k yields the quadratic equation $3a^2 + 5a - 8 = 0$, hence $a = 1$ or $a = -\frac{8}{3}$. Substituting each of these values of a into either of the original equations and solving for k yields $(a, k) = (1, 4)$ or $\left(-\frac{8}{3}, -\frac{149}{48}\right)$. Adding the values of k yields the answer of $\mathbf{43/48}$.

Alternate Solution: In terms of $a = \log_x y$, the two equations become $2a + \frac{5}{a} = 2k - 1$ and $\frac{5a}{2} - \frac{3}{2a} = k - 3$. Eliminate $\frac{1}{a}$ to obtain $31a = 16k - 33$; substitute this into either of the original equations and clear denominators to get $96k^2 - 86k - 1192 = 0$. The sum of the two roots is $86/96 = \mathbf{43/48}$.

Problem 6. Let $W = (0, 0)$, $A = (7, 0)$, $S = (7, 1)$, and $H = (0, 1)$. Compute the number of ways to tile rectangle $WASH$ with triangles of area $1/2$ and vertices at lattice points on the boundary of $WASH$.

Solution 6. Define a *fault line* to be a side of a tile other than its base. Any tiling of $WASH$ can be represented as a sequence of tiles $t_1, t_2, \ldots, t_{14}$, where t_1 has a fault line of $\overline{WH}$, t_{14} has a fault line of $\overline{AS}$, and where t_k and t_{k+1} share a fault line for $1 \leq k \leq 13$. Also note that to determine the position of tile t_{k+1}, it is necessary and sufficient to know the fault line that t_{k+1} shares with t_k, as well as whether the base of t_{k+1} lies on $\overline{WA}$ (abbreviated "B" for "bottom") or on $\overline{SH}$ (abbreviated "T" for "top"). Because rectangle $WASH$ has width 7, precisely 7 of the 14 tiles must have their bases on $\overline{WA}$. Thus any permutation of 7 B's and 7 T's determines a unique tiling $t_1, t_2, \ldots, t_{14}$, and conversely, any tiling $t_1, t_2, \ldots, t_{14}$ corresponds to a unique permutation of 7 B's and 7 T's. Thus the answer is $\binom{14}{7} = \mathbf{3432}$.

Alternate Solution: Let $T(a, b)$ denote the number of ways to triangulate the polygon with vertices at $(0, 0)$, $(b, 0)$, $(a, 1)$, $(0, 1)$, where each triangle has area $1/2$ and vertices at lattice points. The problem is to compute $T(7, 7)$. It is easy to see that $T(a, 0) = T(0, b) = 1$ for all a and b. If a and b

are both positive, then either one of the triangles includes the edge from $(a-1,1)$ to $(b,0)$ or one of the triangles includes the edge from $(a,1)$ to $(b-1,0)$, but not both. (In fact, as soon as there is an edge from $(a,1)$ to $(x,0)$ with $x<b$, there must be edges from $(a,1)$ to $(x',0)$ for all $x \le x' < b$.) If there is an edge from $(a-1,1)$ to $(b,0)$, then the number of ways to complete the triangulation is $T(a-1,b)$; if there is an edge from $(a,1)$ to $(b-1,0)$, then the number of ways to complete the triangulation is $T(a,b-1)$; thus $T(a,b)=T(a-1,b)+T(a,b-1)$. The recursion and the initial conditions describe Pascal's triangle, so $T(a,b)=\binom{a+b}{a}$. In particular, $T(7,7)=\binom{14}{7}=\mathbf{3432}$.

Problem 7. Compute $\sin^2 4^\circ + \sin^2 8^\circ + \sin^2 12^\circ + \cdots + \sin^2 176^\circ$.

Solution 7. Because $\cos 2x = 1-2\sin^2 x$, $\sin^2 x = \frac{1-\cos 2x}{2}$. Thus the desired sum can be rewritten as

$$\frac{1-\cos 8^\circ}{2}+\frac{1-\cos 16^\circ}{2}+\cdots+\frac{1-\cos 352^\circ}{2}=\frac{44}{2}-\frac{1}{2}\left(\cos 8^\circ+\cos 16^\circ+\cdots+\cos 352^\circ\right).$$

If $\alpha=\cos 8^\circ + i\sin 8^\circ$, then α is a primitive 45^{th} root of unity, and $1+\alpha+\alpha^2+\alpha^3+\cdots+\alpha^{44}=0$. Hence $\alpha+\alpha^2+\cdots+\alpha^{44}=-1$, and because the real part of α^n is simply $\cos 8n^\circ$,

$$\cos 8^\circ+\cos 16^\circ+\cdots+\cos 352^\circ=-1.$$

Thus the desired sum is $22-(1/2)(-1)=\mathbf{45/2}$.

Alternate Solution: The problem asks to simplify the sum

$$\sin^2 a+\sin^2 2a+\sin^2 3a+\cdots+\sin^2 na$$

where $a=4^\circ$ and $n=44$. Because $\cos 2x=1-2\sin^2 x$, $\sin^2 x=\frac{1-\cos 2x}{2}$. Thus the desired sum can be rewritten as

$$\frac{1-\cos 2a}{2}+\frac{1-\cos 4a}{2}+\cdots+\frac{1-\cos 2na}{2}=\frac{n}{2}-\frac{1}{2}\left(\cos 2a+\cos 4a+\cdots+\cos 2na\right).$$

Let $Q=\cos 2a+\cos 4a+\cdots+\cos 2na$. By the sum-to-product identity,

$$\begin{aligned}
\sin 3a-\sin a &= 2\cos 2a\sin a,\\
\sin 5a-\sin 3a &= 2\cos 4a\sin a,\\
&\vdots\\
\sin(2n+1)a-\sin(2n-1)a &= 2\cos 2na\sin a.
\end{aligned}$$

Thus

$$\begin{aligned}
Q\cdot 2\sin a &= (\sin 3a-\sin a)+(\sin 5a-\sin 3a)+\cdots+(\sin(2n+1)a-\sin(2n-1)a)\\
&= \sin(2n+1)a-\sin a.
\end{aligned}$$

With $a=4^\circ$ and $n=44$, the difference on the right side becomes $\sin 356^\circ-\sin 4^\circ$; note that the terms in this difference are opposites, because of the symmetry of the unit circle. Hence

$$\begin{aligned}
Q\cdot 2\sin 4^\circ &= -2\sin 4^\circ,\ \text{and}\\
Q &= -1.
\end{aligned}$$

Thus the original sum becomes $44/2-(1/2)(-1)=\mathbf{45/2}$.

Problem 8. Compute the area of the region defined by $x^2 + y^2 \leq |x| + |y|$.

Solution 8. Call the region R, and let R_q be the portion of R in the q^{th} quadrant. Noting that the point (x, y) is in R if and only if $(\pm x, \pm y)$ is in R, it follows that $[R_1] = [R_2] = [R_3] = [R_4]$, and so $[R] = 4[R_1]$. So it suffices to determine $[R_1]$.

In the first quadrant, the boundary equation is just $x^2 + y^2 = x + y \Rightarrow (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{2}$. This equation describes a circle of radius $\frac{\sqrt{2}}{2}$ centered at $(\frac{1}{2}, \frac{1}{2})$. The portion of the circle's interior which is inside the first quadrant can be decomposed into a right isosceles triangle with side length 1 and half a circle of radius $\frac{\sqrt{2}}{2}$. Thus $[R_1] = \frac{1}{2} + \frac{\pi}{4}$, hence $[R] = \mathbf{2 + \boldsymbol{\pi}}$.

Problem 9. The arithmetic sequences $a_1, a_2, a_3, \ldots, a_{20}$ and $b_1, b_2, b_3, \ldots, b_{20}$ consist of 40 distinct positive integers, and $a_{20} + b_{14} = 1000$. Compute the least possible value for $b_{20} + a_{14}$.

Solution 9. Write $a_n = a_1 + r(n-1)$ and $b_n = b_1 + s(n-1)$. Then $a_{20} + b_{14} = a_1 + b_1 + 19r + 13s$, while $b_{20} + a_{14} = a_1 + b_1 + 13r + 19s = a_{20} + b_{14} + 6(s - r)$. Because both sequences consist only of integers, r and s must be integers, so $b_{20} + a_{14} \equiv a_{20} + b_{14} \bmod 6$. Thus the least possible value of $b_{20} + a_{14}$ is 4. If $b_{20} = 3$ and $a_{14} = 1$, then $\{a_n\}$ must be a decreasing sequence (else a_{13} would not be positive) and $a_{20} \leq -5$, which is impossible. The case $b_{20} = a_{14} = 2$ violates the requirement that the terms be distinct, and by reasoning analogous to the first case, $b_{20} = 1, a_{14} = 3$ is also impossible. Hence the sum $b_{20} + a_{14}$ is at least 10. To show that 10 is attainable, make $\{a_n\}$ decreasing and b_{20} as small as possible: set $b_{20} = 1$, $a_{14} = 9$, and $a_n = 23 - n$. Then $a_{20} = 3$, yielding $b_{14} = 997$. Hence $s = \frac{997-1}{14-20} = \frac{996}{-6} = -166$ and $b_1 = 997 - (13)(-166) = 3155$, yielding $b_n = 3155 - 166(n-1)$. Because $b_{20} = 1 \leq a_{20}$ and $b_{19} = 167 \geq a_1$, the sequences $\{b_n\}$ and $\{a_n\}$ are distinct for $1 \leq n \leq 20$, completing the proof. Hence the minimum possible value of $b_{20} + a_{14}$ is **10**. [Note: This solution, which improves on the authors' original solution, is due to Ravi Jagadeesan of Phillips Exeter Academy.]

Problem 10. Compute the ordered triple (x, y, z) representing the farthest lattice point from the origin that satisfies $xy - z^2 = y^2z - x = 14$.

Solution 10. First, eliminate x: $y(y^2z - x) + (xy - z^2) = 14(y+1) \Rightarrow z^2 - y^3z + 14(y+1) = 0$. Viewed as a quadratic in z, this equation implies $z = \frac{y^3 \pm \sqrt{y^6 - 56(y+1)}}{2}$. In order for z to be an integer, the discriminant must be a perfect square. Because $y^6 = (y^3)^2$ and $(y^3 - 1)^2 = y^6 - 2y^3 + 1$, it follows that $|56(y+1)| \geq 2|y^3| - 1$. This inequality only holds for $|y| \leq 5$. Within that range, the only values of y for which $y^6 - 56y - 56$ is a perfect square are -1 and -3. If $y = -1$, then $z = -1$ or $z = 0$. If $y = -3$, then $z = 1$ or $z = -28$. After solving for the respective values of x in the various cases, the four lattice points satisfying the system are $(-15, -1, -1), (-14, -1, 0), (-5, -3, 1)$, and $(-266, -3, -28)$. The farthest solution point from the origin is therefore $\mathbf{(-266, -3, -28)}$.

9 Relay Problems

Relay 1-1 Let $T = (0, 0)$, $N = (2, 0)$, $Y = (6, 6)$, $W = (2, 6)$, and $R = (0, 2)$. Compute the area of pentagon $TNYWR$.

Relay 1-2 Let $T = TNYWR$. The lengths of the sides of a rectangle are the zeroes of the polynomial $x^2 - 3Tx + T^2$. Compute the length of the rectangle's diagonal.

Relay 1-3 Let $T = TNYWR$. Let $w > 0$ be a real number such that T is the area of the region above the x-axis, below the graph of $y = \lceil x \rceil^2$, and between the lines $x = 0$ and $x = w$. Compute $\lceil 2w \rceil$.

Relay 2-1 Compute the least positive integer n such that $\gcd(n^3, n!) \geq 100$.

Relay 2-2 Let $T = TNYWR$. At a party, everyone shakes hands with everyone else exactly once, except Ed, who leaves early. A grand total of $20T$ handshakes take place. Compute the number of people at the party who shook hands with Ed.

Relay 2-3 Let $T = TNYWR$. Given the sequence u_n such that $u_3 = 5$, $u_6 = 89$, and $u_{n+2} = 3u_{n+1} - u_n$ for integers $n \geq 1$, compute u_T.

10 Relay Answers

Answer 1-1 20

Answer 1-2 $20\sqrt{7}$

Answer 1-3 10

Answer 2-1 8

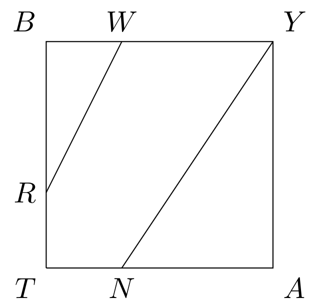
Answer 2-2 7

Answer 2-3 233

11 Relay Solutions

Relay 1-1 Let $T = (0,0)$, $N = (2,0)$, $Y = (6,6)$, $W = (2,6)$, and $R = (0,2)$. Compute the area of pentagon $TNYWR$.

Solution 1-1 Pentagon $TNYWR$ fits inside square $TAYB$, where $A = (6,0)$ and $B = (0,6)$. The region of $TAYB$ not in $TNYWR$ consists of triangles $\triangle NAY$ and $\triangle WBR$, as shown below.



Thus

$$\begin{aligned}[TNYWR] &= [TAYB] - [NAY] - [WBR] \\ &= 6^2 - \frac{1}{2} \cdot 4 \cdot 6 - \frac{1}{2} \cdot 2 \cdot 4 \\ &= \mathbf{20}.\end{aligned}$$

Relay 1-2 Let $T = TNYWR$. The lengths of the sides of a rectangle are the zeroes of the polynomial $x^2 - 3Tx + T^2$. Compute the length of the rectangle's diagonal.

Solution 1-2 Let r and s denote the zeros of the polynomial $x^2 - 3Tx + T^2$. The rectangle's diagonal has length $\sqrt{r^2 + s^2} = \sqrt{(r+s)^2 - 2rs}$. Recall that for a quadratic polynomial $ax^2 + bx + c$, the sum of its zeros is $-b/a$, and the product of its zeros is c/a. In this particular instance, $r + s = 3T$ and $rs = T^2$. Thus the length of the rectangle's diagonal is $\sqrt{9T^2 - 2T^2} = T \cdot \sqrt{7}$. With $T = 20$, the rectangle's diagonal is $\mathbf{20\sqrt{7}}$.

Relay 1-3 Let $T = TNYWR$. Let $w > 0$ be a real number such that T is the area of the region above the x-axis, below the graph of $y = \lceil x \rceil^2$, and between the lines $x = 0$ and $x = w$. Compute $\lceil 2w \rceil$.

Solution 1-3 Write $w = k + \alpha$, where k is an integer, and $0 \le \alpha < 1$. Then

$$T = 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 \cdot \alpha.$$

Computing $\lceil 2w \rceil$ requires computing w to the nearest half-integer. First obtain the integer k. As $\sqrt{7} > 2$, with $T = 20\sqrt{7}$, one obtains $T > 40$. As $1^2 + 2^2 + 3^2 + 4^2 = 30$, it follows that $k \ge 4$. To obtain an upper bound for k, note that $700 < 729$, so $10\sqrt{7} < 27$, and $T = 20\sqrt{7} < 54$. As $1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$, it follows that $4 < w < 5$, and hence $k = 4$.

It now suffices to determine whether or not $\alpha > 0.5$. To this end, one must determine whether $T > 1^2 + 2^2 + 3^2 + 4^2 + 5^2/2 = 42.5$. Indeed, note that $2.5^2 = 6.25 < 7$, so $T > (20)(2.5) = 50$. It follows that $\alpha > 0.5$, so $4.5 < w < 5$. Thus $9 < 2w < 10$, and $\lceil 2w \rceil = \mathbf{10}$.

Alternate Solution: Once it has been determined that $4 < w < 5$, the formula for T yields $1 + 4 + 9 + 16 + 25 \cdot \alpha = 20\sqrt{7}$, hence $\alpha = \frac{4\sqrt{7}-6}{5}$. Thus $2\alpha = \frac{8\sqrt{7}-12}{5} = \frac{\sqrt{448}-12}{5} > \frac{21-12}{5} = 1.8$. Because $2w = 2k + 2\alpha$, it follows that $\lceil 2w \rceil = \lceil 8 + 2\alpha \rceil = \mathbf{10}$, because $1.8 < 2\alpha < 2$.

Relay 2-1 Compute the least positive integer n such that $\gcd(n^3, n!) \geq 100$.

Solution 2-1 Note that if p is prime, then $\gcd(p^3, p!) = p$. A good strategy is to look for values of n with several (not necessarily distinct) prime factors so that n^3 and $n!$ will have many factors in common. For example, if $n = 6$, $n^3 = 216 = 2^3 \cdot 3^3$ and $n! = 720 = 2^4 \cdot 3^2 \cdot 5$, so $\gcd(216, 720) = 2^3 \cdot 3^2 = 72$. Because 7 is prime, try $n = 8$. Notice that $8^3 = 2^9$ while $8! = 2^7 \cdot 3^2 \cdot 5 \cdot 7$. Thus $\gcd(512, 8!) = 2^7 = 128 > 100$, hence the smallest value of n is **8**.

Relay 2-2 Let $T = TNYWR$. At a party, everyone shakes hands with everyone else exactly once, except Ed, who leaves early. A grand total of $20T$ handshakes take place. Compute the number of people at the party who shook hands with Ed.

Solution 2-2 If there were n people at the party, including Ed, and if Ed had not left early, there would have been $\binom{n}{2}$ handshakes. Because Ed left early, the number of handshakes is strictly less than that, but greater than $\binom{n-1}{2}$ (everyone besides Ed shook everyone else's hand). So find the least number n such that $\binom{n}{2} \geq 160$. The least such n is 19, because $\binom{18}{2} = 153$ and $\binom{19}{2} = 171$. Therefore there were 19 people at the party. However, $171 - 160 = 11$ handshakes never took place. Therefore the number of people who shook hands with Ed is $19 - 11 - 1 = \mathbf{7}$.

Relay 2-3 Let $T = TNYWR$. Given the sequence u_n such that $u_3 = 5$, $u_6 = 89$, and $u_{n+2} = 3u_{n+1} - u_n$ for integers $n \geq 1$, compute u_T.

Solution 2-3 By the recursive definition, notice that $u_6 = 89 = 3u_5 - u_4$ and $u_5 = 3u_4 - u_3 = 3u_4 - 5$. This is a linear system of equations. Write $3u_5 - u_4 = 89$ and $-3u_5 + 9u_4 = 15$ and add to obtain $u_4 = 13$. Now apply the recursive definition to obtain $u_5 = 34$ and $u_7 = \mathbf{233}$.

Alternate Solution: Notice that the given values are both Fibonacci numbers, and that in the Fibonacci sequence, $f_1 = f_2 = 1$, $f_5 = 5$, and $f_{11} = 89$. That is, 5 and 89 are six terms apart in the Fibonacci sequence, and only three terms apart in the given sequence. This relationship is not a coincidence: alternating terms in the Fibonacci sequence satisfy the given recurrence relation for the sequence $\{u_n\}$, that is, $f_{n+4} = 3f_{n+2} - f_n$. Proof: if $f_n = a$ and $f_{n+1} = b$, then $f_{n+2} = a + b$, $f_{n+3} = a + 2b$, and $f_{n+4} = 2a + 3b = 3(a + b) - b = 3f_{n+2} - f_n$. To compute the final result, continue out the Fibonacci sequence to obtain $f_{12} = 144$ and $u_7 = f_{13} = \mathbf{233}$.