**CT ARML Team, 2023**

**Team Selection Test 2**

1. Andy is taking free throws. His *shooting accuracy* is the percentage of throws on which he is successful. After taking $n$ free throws his shooting accuracy is 65%. He then makes $k$ successful throws, and his shooting accuracy after $n + k$ throws is 70%. Compute the minimum possible value of $k$.
[Answer: 10]

2. A particle starts moving on the number line at time $t = 0$. Its position at time $t$ is $x = (t - 2022)^2 - 2022(t - 2022) - 2023$. Compute the number of positive integer values of $t$ at which the particle lies in the negative half of the number line (strictly to the left of 0).
[Answer: 2023]

3. For each positive integer $n$, let $x_n + iy_n = (1 + i\sqrt{3})^n$, where $x_n$ and $y_n$ are real. Suppose that $x_{19}y_{91} + x_{91}y_{19} = 2^k\sqrt{3}$. Compute $k$.
[Answer: 109]

4. Let $\Omega$ be a semicircle with diameter $AB = 10$. There exist points $C$ and $D$ on $\Omega$ (meaning that $C$ and $D$ lie on the arc $\widehat{AB}$) such that $\overline{AC}$ and $\overline{BD}$ intersect at a point $E$ in the interior of the semicircle, with $AE = 6$ and $CE = 2$. Then $AD^2 = \frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Compute $m + n$.
[Answer: 167]

5. Find the number of four-digit numbers $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$ with distinct, nonzero digits such that $A < B$ and $C < D$.
[Answer: 756]

6. For real numbers $x$, let $f(x) = 16x^3 - 21x$. Given that $\theta$ is an angle satisfying $f(\sin\theta) = f(\cos\theta)$, the sum of all possible values of $\tan^2\theta$ is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Compute $m + n$.
[Answer: 256]

7. Let $f$ and $g$ be quadratic polynomials. Suppose that $f$ has zeros 2 and 7, $g$ has zeros 1 and 8, and $f - g$ has zeros 4 and 5. Compute the product of the zeros of the polynomial $f + g$.
[Answer: 12]

8. The base-10 fraction $\frac{20}{23}$ has period $a$ when written (as a repeating decimal) in base 2023. Compute the value of $a$.
(Note: The *period* of a repeating decimal is the number of digits after which the pattern repeats. For example, the period of $0.476476476\ldots$ is 3.)
[Answer: 2]

9. Find the greatest possible integer $n$ such that there is a unique positive integer $m$ that satisfies

$$\frac{1}{10} \leq \frac{m}{n} \leq \frac{1}{9}.$$

[Answer: 161]

10. James has 16 identical apples. James will distribute these apples among his six friends: Albert, Bevin, Camila, Davon, Emma, and Frank. Each of the first three of these people will receive a number of apples that is a multiple of 3 (possibly 0), and each of the last three will receive a number of apples that is not a multiple of 3. Compute the number of possible ways in which James can distribute his apples.
[Answer: 378]

**CT ARML Team, 2023**

**2023 CT ARML Team Selection Test 2 Solution**

1. Andy is taking free throws. His *shooting accuracy* is the percentage of throws on which he is successful. After taking $n$ free throws his shooting accuracy is 65%. He then makes $k$ successful throws, and his shooting accuracy after $n+k$ throws is 70%. Compute the minimum possible value of $k$.

**Solution:**.

$$\frac{0.65\,n+k}{n+k}=0.7$$

Solve to get $n=6k$. We will need to make sure $0.65n=3.9k$ is an integer, and the minimum value of $k$ to make it an integer is $\boxed{10}$.

2. A particle starts moving on the number line at time $t=0$. Its position at time $t$ is $x=(t-2022)^2-2022(t-2022)-2023$. Compute the number of positive integer values of $t$ at which the particle lies in the negative half of the number line (strictly to the left of 0).

**Solution:**

Let $s=t-2022$ and substitute it to the equation, we get

$$x=s^2-2022s-2023=(s-2023)(s+1)$$

In order to make $x<0, s\in(-1,2023), thus\ t\in(2021,4045)$.

The number of positive integer values of t is $4045-2021-1=\boxed{2023.}$

3. For each positive integer $n$, let $x_n+iy_n=(1+i\sqrt{3})^n$, where $x_n$ and $y_n$ are real. Suppose that $x_{19}y_{91}+x_{91}y_{19}=2^k\sqrt{3}$. Compute $k$.

**Solution:**

We want to first convert $(1+\sqrt{3}i)$ to its exponential form $re^{i\theta}$.

$$r=\sqrt{1+3}=2$$

$$\tan\theta=\sqrt{3}\ and\ we\ can\ take\ \theta=\frac{\pi}{3}$$

So $\left(1+\sqrt{3}i\right)=2e^{\pi i/3}$.

$$x_{19}+iy_{19}=(2e^{\frac{\pi i}{3}})^{19}$$

$$x_{91}+iy_{91}=(2e^{\frac{\pi i}{3}})^{91}$$

We can find $x_{19}, y_{91,}x_{91}, and\ y_{19}$ individually from above equations. However, we can tell $x_{19}y_{91}+x_{91}y_{19}\ is\ actually\ the\ imaginary\ part\ of\ (x_{19}+iy_{19})(\,x_{91}+iy_{91})$.

$$(x_{19}+iy_{19})(\,x_{91}+iy_{91})=\left(2e^{\frac{\pi i}{3}}\right)^{19}\left(2e^{\frac{\pi i}{3}}\right)^{91}=\left(2e^{\frac{\pi i}{3}}\right)^{110}$$

$$= 2^{110}3^{110\pi i/3} = 2^{110}e^{2\pi i/3} = 2^{110}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = -2^{109} + 2^{109}\sqrt{3}i$$

The answer is $\boxed{109}$.

4. Let $\Omega$ be a semicircle with diameter $AB = 10$. There exist points $C$ and $D$ on $\Omega$ (meaning that $C$ and $D$ lie on the arc $\overset{\frown}{AB}$) such that $\overline{AC}$ and $\overline{BD}$ intersect at a point $E$ in the interior of the semicircle, with $AE = 6$ and $CE = 2$. Then $AD^2 = \frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Compute $m + n$.

**Solution:**

Since AB is a diameter, we know both triangle ACB and ADB are right triangles.

$$AB = 10, AC = AE + CE = 8, so\ BC = 6$$

$$\because \varDelta ADE \sim \varDelta BCE, \frac{AD}{DE} = \frac{BC}{CE} = 3$$

Let DE = $x$, AD = $3x$.

$$\because \varDelta ADE \sim \varDelta BCE, \frac{AE}{DE} = \frac{BE}{CE}, \frac{6}{x} = \frac{BE}{2}$$

$BE = \frac{12}{x}$

$$AB^2 = AD^2 + BD^2$$

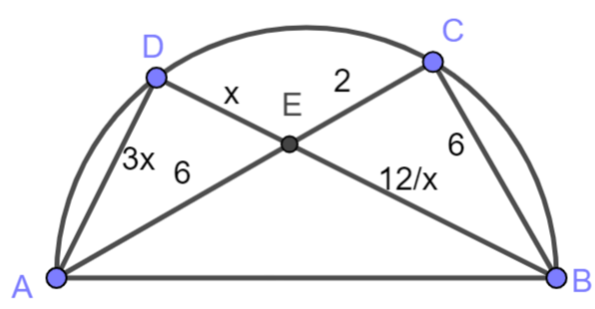
$$100 = 9x^2 + (x + \frac{12}{x})^2$$

$$5x^2 + \frac{72}{x^2} - 38 = 0$$

*Solve to obtain* $x^2 = \frac{18}{5}$

$$AD^2 = 9x^2 = \frac{162}{5}$$

The answer is $\boxed{167}$.




5. Find the number of four-digit numbers $\underline{A}\ \underline{B}\ \underline{C}\ \underline{D}$ with distinct, nonzero digits such that $A < B$ and $C < D$.

**Solution:**

Consider the two pairs separately. For the pair AB, there are $\binom{9}{2} = 36$ ways to pick two digits and only one way to arrange them. For the pair CD, there are $\binom{7}{2} = 21$ ways to pick two digits and again only way to arrange them. So there are a total of 36 x 21 = $\boxed{756}$ possible numbers.

6. For real numbers $x$, let $f(x) = 16x^3 - 21x$. Given that $\theta$ is an angle satisfying $f(\sin\theta) = f(\cos\theta)$, the sum of all possible values of $\tan^2\theta$ is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Compute $m + n$.

**Solution:**

$$f(sin\theta) = 16sin^3\theta - 21sin\theta = f(cos\theta) = 16cos^3\theta - 21cos\theta$$

Rearrange the above equation we have

$$16(sin^3\theta - cos^3\theta) = 21(sin\theta - cos\theta)$$

Case 1: $sin\theta - cos\theta = 0; tan\theta = 1$

Case 2: $sin\theta - cos\theta \neq 0$; we can divide $(sin\theta - cos\theta)$ on both side to get

$$16(sin^2\theta + sin\theta cos\theta + cos^2\theta) = 21$$

$$sin\theta cos\theta = \frac{5}{16}$$

We can consider $sin\theta$ and $cos\theta$ as the two roots of the below equation

$$\begin{cases} x^2 + y^2 = 1 \\ xy = \dfrac{5}{16} \end{cases}$$

Thus the two possible values of $tan\theta\ are\ \frac{x}{y}\ and\ \frac{y}{x}$, and so the sum of the values $tan^2\theta$ is

$$\frac{x^2}{y^2} + \frac{y^2}{x^2} = \frac{(x^2 + y^2)^2}{(xy)^2} - 2 = \frac{256}{25} - 2 = \frac{206}{25}$$

Adding the case 1 and case 2 together gives the sum as $\frac{231}{25}$ so the answer is $\boxed{256}$.

7. Let $f$ and $g$ be quadratic polynomials. Suppose that $f$ has zeros 2 and 7, $g$ has zeros 1 and 8, and $f - g$ has zeros 4 and 5. Compute the product of the zeros of the polynomial $f + g$.

**Solution:**

$Let\ f = a(x-2)(x-7)\ and\ g = b(x-1)(x-8)$ where $a$ and $b$ are two nonzero constants. Based on the statement $f - g$ has zeroes 4 and 5, we can plug either $x = 4\ or\ x = 5$, which gives $-6a = -12b, or\ a = 2b$. WLOG, assume $a = 1\ and\ b = 2$, then

$$f + g = (x-2)(x-7) + 2(x-1)(x-8) = 3x^2 - 27x + 36$$

By Vieta's theorem, the product of the two roots of $f + g$ is $\boxed{12}$.

8. The base-10 fraction $\frac{20}{23}$ has period $a$ when written (as a repeating decimal) in base 2023. Compute the value of $a$.
(Note: The *period* of a repeating decimal is the number of digits after which the pattern repeats. For example, the period of 0.476476476… is 3.)

**Solution 1:**

We can convert the base-10 fraction to base 2023 decimal step by step.

$$20 \times 2023 = 40460 = 23 \times (1759) + 3$$

So the first digit after the decimal point is $1759_{2023}$ and the remainder is 3

$$3 \times 2023 = 6069 = 23 \times (263) + 20$$

The second digit after the decimal point is $263_{2023}$. Since the remainder is 20, this is the same as the numerator of the original fraction so the period is $\boxed{2}$.

**Solution 2:**

*a* is the smallest integer such that $2023^a - 1$ is a multiple of 23, which is

$$2023^a \equiv 1(mod\ 23)$$

Since $2023 \equiv -1(mod\ 23), 2023^2 \equiv 1(mod\ 23), so\ a = 2$. The period is $\boxed{2}$.

9. Find the greatest possible integer $n$ such that there is a unique positive integer $m$ that satisfies

$$\frac{1}{10} \le \frac{m}{n} \le \frac{1}{9}.$$

**Solution:**

$$\frac{1}{10} \le \frac{m}{n} \le \frac{1}{9}$$
$$9m \le n \le 10m$$

Since *m* is a unique positive integer, this inequality cannot be satisfied by a greater value of *m*, so $n < 9(m+1)$. Similarly we can get $n > 10(m-1)$.

So $10m - 10 < n < 9m + 9$. Since this inequality is strict, this means the interval from $10m - 10\ and\ 9m + 9$ must have width at least 2 to allow intermediate value *n*,

So $9m + 9 - (10m - 10) \ge 2, or\ m \le 17$.

Thus the maximum *n* occurs when $m = 17, giving\ 9m + 9 = 162\ and\ 10m - 10 = 160, so$ *n* = $\boxed{161}$

10. James has 16 identical apples. James will distribute these apples among his six friends: Albert, Bevin, Camila, Davon, Emma, and Frank. Each of the first three of these people will receive a number of apples that is a multiple of 3 (possibly 0), and each of the last three will receive a number of apples that is not a multiple of 3. Compute the number of possible ways in which James can distribute his apples.

**Solution:**

Let Albert, Bevin, Camila, Davon, Emma, and Frank get *a, b, c, d, e, f* apples. Since $16 \equiv 1\ (mod\ 3)$, we can conclude one of *d, e, f* has to be 2 *(mod 3)*, while the other two are 1 *(mod 3)*.

There are 3 ways to pick which one is 2 *(mod 3)*. Assume it is *d* and we can write

$$a = 3a', b = 3b', c = 3c', d = 3d' + 2, e = 3e' + 1, f = 3f' + 1$$

Note $a', b', c', d', e' and\ f' are\ all\ nonnegative\ integers$.

Since $a + b + c + d + e + f = 16$, we can substitute and rearrange to obtain

$$a' + b' + c' + d' + e' + f' = 4$$

This then becomes a Stars and Bars problem, and there are $\binom{9}{4} = 126$ ways to distribute 4 apples among 6 people. Multiple by 3 to count for the apple distribution among *d*, *e*, *f*, we get $\boxed{378}$ as answer.

## 4 Power Question 2014: Power of Potlucks

**Instructions:** The power question is worth 50 points; each part's point value is given in brackets next to the part. To receive full credit, the presentation must be legible, orderly, clear, and concise. If a problem says "list" or "compute," you need not justify your answer. If a problem says "determine," "find," or "show," then you must show your work or explain your reasoning to receive full credit, although such explanations do not have to be lengthy. If a problem says "justify" or "prove," then you must prove your answer rigorously. Even if not proved, earlier numbered items may be used in solutions to later numbered items, but not vice versa. Pages submitted for credit should be NUMBERED IN CONSECUTIVE ORDER AT THE TOP OF EACH PAGE in what your team considers to be proper sequential order. PLEASE WRITE ON ONLY ONE SIDE OF THE ANSWER PAPERS. Put the TEAM NUMBER (not the team name) on the cover sheet used as the first page of the papers submitted. Do not identify the team in any other way.

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In each town in ARMLandia, the residents have formed groups, which meet each week to share math problems and enjoy each others' company over a potluck-style dinner. Each town resident belongs to exactly one group. Every week, each resident is required to make one dish and to bring it to his/her group.

It so happens that each resident knows how to make precisely two dishes. Moreover, no two residents of a town know how to make the same pair of dishes. Shown below are two example towns. In the left column are the names of the town's residents. Adjacent to each name is the list of dishes that the corresponding resident knows how to make.

ARMLton

| Resident | Dishes |
|---|---|
| Paul | pie, turkey |
| Arnold | pie, salad |
| Kelly | salad, broth |

ARMLville

| Resident | Dishes |
|---|---|
| Sally | steak, calzones |
| Ross | calzones, pancakes |
| David | steak, pancakes |

The *population* of a town $T$, denoted $\operatorname{pop}(T)$, is the number of residents of $T$. Formally, the town itself is simply the set of its residents, denoted by $\{r_1, \ldots, r_{\operatorname{pop}(T)}\}$ unless otherwise specified. The set of dishes that the residents of $T$ collectively know how to make is denoted $\operatorname{dish}(T)$. For example, in the town of ARMLton described above, $\operatorname{pop}(\text{ARMLton}) = 3$, and $\operatorname{dish}(\text{ARMLton}) = \{\text{pie, turkey, salad, broth}\}$.

A town $T$ is called *full* if for every pair of dishes in $\operatorname{dish}(T)$, there is exactly one resident in $T$ who knows how to make those two dishes. In the examples above, ARMLville is a full town, but ARMLton is not, because (for example) nobody in ARMLton knows how to make both turkey and salad.

Denote by $\mathcal{F}_d$ a full town in which collectively the residents know how to make $d$ dishes. That is, $|\operatorname{dish}(\mathcal{F}_d)| = d$.

**1a.** Compute $\operatorname{pop}(\mathcal{F}_{17})$. [1 pt]

**1b.** Let $n = \operatorname{pop}(\mathcal{F}_d)$. In terms of $n$, compute $d$. [1 pt]

**1c.** Let $T$ be a full town and let $D \in \operatorname{dish}(T)$. Let $T'$ be the town consisting of all residents of $T$ who do not know how to make $D$. Prove that $T'$ is full. [2 pts]

In order to avoid the embarrassing situation where two people bring the same dish to a group dinner, if two people know how to make a common dish, they are forbidden from participating in the same group meeting. Formally, a *group assignment* on $T$ is a function $f : T \to \{1, 2, \ldots, k\}$, satisfying the condition that if $f(r_i) = f(r_j)$ for $i \neq j$, then $r_i$ and $r_j$ do not know any of the same recipes. The *group number* of a town $T$, denoted $\operatorname{gr}(T)$, is the least positive integer $k$ for which there exists a group assignment on $T$.

For example, consider once again the town of ARMLton. A valid group assignment would be $f(\text{Paul}) = f(\text{Kelly}) = 1$ and $f(\text{Arnold}) = 2$. The function which gives the value 1 to each resident of ARMLton is **not** a group assignment, because Paul and Arnold must be assigned to different groups.

**2a.** Show that $\text{gr}(\text{ARMLton}) = 2$. [1 pt]

**2b.** Show that $\text{gr}(\text{ARMLville}) = 3$. [1 pt]

**3a.** Show that $\text{gr}(\mathcal{F}_4) = 3$. [1 pt]

**3b.** Show that $\text{gr}(\mathcal{F}_5) = 5$. [2 pts]

**3c.** Show that $\text{gr}(\mathcal{F}_6) = 5$. [2 pts]

**4.** Prove that the sequence $\text{gr}(\mathcal{F}_2), \text{gr}(\mathcal{F}_3), \text{gr}(\mathcal{F}_4), \ldots$ is a non-decreasing sequence. [2 pts]

For a dish $D$, a resident is called a *$D$-chef* if he or she knows how to make the dish $D$. Define $\text{chef}_T(D)$ to be the set of residents in $T$ who are $D$-chefs. For example, in ARMLville, David is a steak-chef and a pancakes-chef. Further, $\text{chef}_{\text{ARMLville}}(\text{steak}) = \{\text{Sally, David}\}$.

**5.** Prove that

$$\sum_{D \in \text{dish}(T)} |\text{chef}_T(D)| = 2\,\text{pop}(T).$$ [2 pts]

**6.** Show that for any town $T$ and any $D \in \text{dish}(T)$, $\text{gr}(T) \geq |\text{chef}_T(D)|$. [2 pts]

If $\text{gr}(T) = |\text{chef}_T(D)|$ for some $D \in \text{dish}(T)$, then $T$ is called *homogeneous*. If $\text{gr}(T) > |\text{chef}_T(D)|$ for each dish $D \in \text{dish}(T)$, then $T$ is called *heterogeneous*. For example, ARMLton is homogeneous, because $\text{gr}(\text{ARMLton}) = 2$ and exactly two chefs make pie, but ARMLville is heterogeneous, because even though each dish is only cooked by two chefs, $\text{gr}(\text{ARMLville}) = 3$.

**7.** For $n = 5, 6,$ and $7$, find a heterogeneous town $T$ of population 5 for which $|\text{dish}(T)| = n$. [3 pts]

A *resident cycle* is a sequence of distinct residents $r_1, \ldots, r_n$ such that for each $1 \leq i \leq n-1$, the residents $r_i$ and $r_{i+1}$ know how to make a common dish, residents $r_n$ and $r_1$ know how to make a common dish, and no other pair of residents $r_i$ and $r_j$, $1 \leq i, j \leq n$ know how to make a common dish. Two resident cycles are *indistinguishable* if they contain the same residents (in any order), and distinguishable otherwise. For example, if $r_1, r_2, r_3, r_4$ is a resident cycle, then $r_2, r_1, r_4, r_3$ and $r_3, r_2, r_1, r_4$ are indistinguishable resident cycles.

**8a.** Compute the number of distinguishable resident cycles of length 6 in $\mathcal{F}_8$. [1 pt]

**8b.** In terms of $k$ and $d$, find the number of distinguishable resident cycles of length $k$ in $\mathcal{F}_d$. [1 pt]

**9.** Let $T$ be a town with at least two residents that has a single resident cycle that contains every resident. Prove that $T$ is homogeneous if and only if $\text{pop}(T)$ is even. [3 pts]

**10.** Let $T$ be a town such that, for each $D \in \text{dish}(T)$, $|\text{chef}_T(D)| = 2$.

- **a.** Prove that there are finitely many resident cycles $C_1, C_2, \ldots, C_j$ in $T$ so that each resident belongs to exactly one of the $C_i$. [3 pts]
- **b.** Prove that if $\text{pop}(T)$ is odd, then $T$ is heterogeneous. [3 pts]

**11.** Let $T$ be a town such that, for each $D \in \text{dish}(T)$, $|\text{chef}_T(D)| = 3$.

- **a.** Either find such a town $T$ for which $|\text{dish}(T)|$ is odd, or show that no such town exists. [2 pts]
- **b.** Prove that if $T$ contains a resident cycle such that for every dish $D \in \text{dish}(T)$, there exists a chef in the cycle that can prepare $D$, then $\text{gr}(T) = 3$. [3 pts]

**12.** Let $k$ be a positive integer, and let $T$ be a town in which $|\text{chef}_T(D)| = k$ for every dish $D \in \text{dish}(T)$. Suppose further that $|\text{dish}(T)|$ is odd.

- **a.** Show that $k$ is even. [1 pt]
- **b.** Prove the following: for every group in $T$, there is some dish $D \in \text{dish}(T)$ such that no one in the group is a $D$-chef. [3 pts]
- **c.** Prove that $\text{gr}(T) > k$. [3 pts]

**13a.** For each odd positive integer $d \geq 3$, prove that $\text{gr}(\mathcal{F}_d) = d$. [3 pts]

**13b.** For each even positive integer $d$, prove that $\text{gr}(\mathcal{F}_d) = d - 1$. [4 pts]

# 5 Solutions to Power Question

**1.** **a.** There are $\binom{17}{2} = 136$ possible pairs of dishes, so $\mathcal{F}_{17}$ must have 136 people.

**b.** With $d$ dishes there are $\binom{d}{2} = \frac{d^2-d}{2}$ possible pairs, so $n = \frac{d^2-d}{2}$. Then $2n = d^2 - d$, or $d^2 - d - 2n = 0$. Using the quadratic formula yields $d = \frac{1+\sqrt{1+8n}}{2}$ (ignoring the negative value).

**c.** The town $T'$ consists of all residents of $T$ who do not know how to make $D$. Because $T$ is full, every pair of dishes $\{d_i, d_j\}$ in dish$(T)$ can be made by some resident $r_{ij}$ in $T$. If $d_i \neq D$ and $d_j \neq D$, then $r_{ij} \in T'$. So every pair of dishes in dish$(T) \setminus \{D\}$ can be made by some resident of $T'$. Hence $T'$ is full.

**2.** **a.** Paul and Arnold cannot be in the same group, because they both make pie, and Arnold and Kelly cannot be in the same group, because they both make salad. Hence there must be at least two groups. But Paul and Kelly make none of the same dishes, so they can be in the same group. Thus a valid group assignment is

$$\begin{aligned} \text{Paul} &\mapsto 1 \\ \text{Kelly} &\mapsto 1 \\ \text{Arnold} &\mapsto 2. \end{aligned}$$

Hence gr(ARMLton) = 2.

**b.** Sally and Ross both make calzones, Ross and David both make pancakes, and Sally and David both make steak. So no two of these people can be in the same group, and gr(ARMLville) = 3.

**3.** **a.** Let the dishes be $d_1, d_2, d_3, d_4$ and let resident $r_{ij}$ make dishes $d_i$ and $d_j$, where $i < j$. There are six pairs of dishes, which can be divided into nonoverlapping pairs in three ways: $\{1,2\}$ and $\{3,4\}$, $\{1,3\}$ and $\{2,4\}$, and $\{1,4\}$ and $\{2,3\}$. Hence the assignment $r_{12}, r_{34} \mapsto 1$, $r_{13}, r_{24} \mapsto 2$, and $r_{14}, r_{23} \mapsto 3$ is valid, hence $\text{gr}(\mathcal{F}_4) = 3$.

**b.** First, $\text{gr}(\mathcal{F}_5) \geq 5$: there are $\binom{5}{2} = 10$ people in $\mathcal{F}_5$, and because each person cooks two different dishes, any valid group of three people would require there to be six different dishes—yet there are only five. So each group can have at most two people. A valid assignment using five groups is shown below.

| Residents | Group |
|---|---|
| $r_{12}, r_{35}$ | 1 |
| $r_{13}, r_{45}$ | 2 |
| $r_{14}, r_{23}$ | 3 |
| $r_{15}, r_{24}$ | 4 |
| $r_{25}, r_{34}$ | 5 |

**c.** Now there are $\binom{6}{2} = 15$ people, but there are six different dishes, so it is possible (if done carefully) to place three people in a group. Because four people in a single group would require there to be eight different dishes, no group can have more than three people, and so $15/3 = 5$ groups is minimal. (Alternatively, there are five different residents who can cook dish $d_1$, and no two of these can be in the same group, so there must be at least five groups.) The assignment below attains that minimum.

| Residents | Group |
|---|---|
| $r_{12}, r_{34}, r_{56}$ | 1 |
| $r_{13}, r_{25}, r_{46}$ | 2 |
| $r_{14}, r_{26}, r_{35}$ | 3 |
| $r_{15}, r_{24}, r_{36}$ | 4 |
| $r_{16}, r_{23}, r_{45}$ | 5 |

**4.** Pick some $n \geq 2$ and a full town $\mathcal{F}_n$ whose residents prepare dishes $d_1, \ldots, d_n$, and let $\mathrm{gr}(\mathcal{F}_n) = k$. Suppose that $f_n : \mathcal{F}_n \to \{1, 2, \ldots, k\}$ is a valid group assignment for $\mathcal{F}_n$. Then remove from $\mathcal{F}_n$ all residents who prepare dish $d_n$; by problem 1c, this operation yields the full town $\mathcal{F}_{n-1}$. Define $f_{n-1}(r) = f_n(r)$ for each remaining resident $r$ in $\mathcal{F}_n$. If $r$ and $s$ are two (remaining) residents who prepare a common dish, then $f_n(r) \neq f_n(s)$, because $f_n$ was a valid group assignment. Hence $f_{n-1}(r) \neq f_{n-1}(s)$ by construction of $f_{n-1}$. Therefore $f_{n-1}$ is a valid group assignment on $\mathcal{F}_{n-1}$, and the set of groups to which the residents of $\mathcal{F}_{n-1}$ are assigned is a (not necessarily proper) subset of $\{1, 2, \ldots, k\}$. Thus $\mathrm{gr}(\mathcal{F}_{n-1})$ is at most $k$, which implies the desired result.

**5.** Because each chef knows how to prepare exactly two dishes, and no two chefs know how to prepare the same two dishes, each chef is counted exactly twice in the sum $\Sigma\,|\mathrm{chef}_T(D)|$. More formally, consider the set of "resident-dish pairs":

$$S = \{(r, D) \in T \times \mathrm{dish}(T) \mid r \text{ makes } D\}.$$

Count $|S|$ in two different ways. First, every dish $D$ is made by $|\mathrm{chef}_T(D)|$ residents of $T$, so

$$|S| = \sum_{D \in \mathrm{dish}(T)} |\mathrm{chef}_T(D)|\,.$$

Second, each resident knows how to make exactly two dishes, so

$$|S| = \sum_{r \in T} 2 = 2\,\mathrm{pop}(T).$$

**6.** Let $D \in \mathrm{dish}(T)$. Suppose that $f$ is a valid group assignment on $T$. Then for $r, s \in \mathrm{chef}_T(D)$, if $r \neq s$, it follows that $f(r) \neq f(s)$. Hence there must be at least $|\mathrm{chef}_T(D)|$ distinct groups in the range of $f$, i.e., $\mathrm{gr}(T) \geq |\mathrm{chef}_T(D)|$.

**7.** For $n = 5$, this result is attained as follows:

| Resident | Dishes |
|---|---|
| Amy | $d_1, d_2$ |
| Benton | $d_2, d_3$ |
| Carol | $d_3, d_4$ |
| Devin | $d_4, d_5$ |
| Emma | $d_5, d_1$ |

For each dish $D$, note that $\text{chef}_T(D) = 2$. But $\text{gr}(T) > 2$, because if $T$ had at most two groups, at least one of them would contain three people, and choosing any three people will result in a common dish that two of them can cook. Hence $T$ is heterogeneous.

For $n \geq 6$, it suffices to assign dishes to residents so that there are three people who must be in different groups and that no dish is cooked by more than two people, which guarantees that $\text{gr}(T) \geq 3$ and $\text{chef}_T(D) \leq 2$ for each dish $D$.

| Resident | Dishes |
|---|---|
| Amy | $d_1, d_2$ |
| Benton | $d_1, d_3$ |
| Carol | $d_2, d_3$ |
| Devin | $d_4, d_5$ |
| Emma | $d_5, d_6$ |

Note that Devin's and Emma's dishes are actually irrelevant to the situation, so long as they do not cook any of $d_1, d_2, d_3$, which already have two chefs each. Thus we can adjust this setup for $n = 7$ by setting Devin's dishes as $d_4, d_5$ and Emma's dishes as $d_6, d_7$. (In this last case, Devin and Emma are extremely compatible: they can both be put in a group with anyone else in the town!)

**8.** **a.** Because the town is full, each pair of dishes is cooked by exactly one resident, so it is simplest to identify residents by the pairs of dishes they cook. Suppose the first resident cooks $(d_1, d_2)$, the second resident $(d_2, d_3)$, the third resident $(d_3, d_4)$, and so on, until the sixth resident, who cooks $(d_6, d_1)$. Then there are 8 choices for $d_1$ and 7 choices for $d_2$. There are only 6 choices for $d_3$, because $d_3 \neq d_1$ (otherwise two residents would cook the same pair of dishes). For $k > 3$, the requirement that no two intermediate residents cook the same dishes implies that $d_{k+1}$ cannot equal any of $d_1, \ldots, d_{k-1}$, and of course $d_k$ and $d_{k+1}$ must be distinct dishes. Hence there are $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 20{,}160$ six-person resident cycles, not accounting for different starting points in the cycle and the two different directions to go around the cycle. Taking these into account, there are $20{,}160/(6 \cdot 2) = 1{,}680$ distinguishable resident cycles.

**b.** Using the logic from 8a, there are $d(d-1)\cdots(d-k+1)$ choices for $d_1, d_2, \ldots, d_k$. To account for indistinguishable cycles, divide by $k$ possible starting points and 2 possible directions, yielding $\frac{d(d-1)\cdots(d-k+1)}{2k}$ or $\frac{d!}{2k(d-k)!}$ distinguishable resident cycles.

**9.** Note that for every $D \in \text{dish}(T), \text{chef}_T(D) \leq 2$, because otherwise, $r_1, r_2, \ldots, r_n$ could not be a resident cycle. Without loss of generality, assume the cycle is $r_1, r_2, \ldots, r_n$. If $n$ is even, assign resident $r_i$ to group 1 if $i$ is odd, and to group 2 if $i$ is even. This is a valid group assignment, because the only pairs of residents who cook the same dish are $(r_i, r_{i+1})$ for $i = 1, 2, \ldots, n-1$ and $(r_n, r_1)$. In each case, the residents are assigned to different groups. This proves $\text{gr}(T) = 2$, so $T$ is homogeneous. On the other hand, if $n$ is odd, suppose for the sake of contradiction that there are only two groups. Then either $r_1$ and $r_n$ are in the same group, or for some $i$, $r_i$ and $r_{i+1}$ are in the same group. In either case, two residents in the same group share a dish, contradicting the requirement that no members of a group have a common dish. Hence $\text{gr}(T) \geq 3$ when $n$ is odd, making $T$ heterogeneous.

**10.** **a.** First note that the condition $|\text{chef}_T(D)| = 2$ for all $D$ implies that $\text{pop}(T) = |\text{dish}(T)|$, using the equation from problem 5. So for the town in question, the population of the town equals the number of dishes in the town. Because no two chefs cook the same pair of dishes, it is impossible for such a town to have exactly two residents, and because each dish is cooked by exactly two chefs, it is impossible for such a town to have only one resident.

The claim is true for towns of three residents satisfying the conditions: such towns must have one resident who cooks dishes $d_1$ and $d_2$, one resident who cooks dishes $d_2$ and $d_3$, and one resident who cooks dishes $d_3$ and $d_1$, and those three residents form a cycle. So proceed by (modified) strong induction: assume that for some $n > 3$ and for all positive integers $k$ such that $3 \leq k < n$, every town $T$ with $k$ residents and $|\text{chef}_T(D)| = 2$ for all $D \in \text{dish}(T)$ can be divided into a finite number of resident cycles such that each resident belongs to exactly one of the cycles. Let $T_n$ be a town of $n$ residents, and arbitrarily pick resident $r_1$ and dishes $d_1$ and $d_2$ cooked by $r_1$. Then there is exactly one other resident $r_2$ who also cooks $d_2$ (because $|\text{chef}_{T_n}(d_2)| = 2$). But $r_2$ also cooks another dish, $d_3$, which is cooked by another resident, $r_3$. Continuing in this fashion, there can be only two outcomes: either the process exhausts all the residents of $T_n$, or there exists some resident $r_m$, $m < n$, who cooks the same dishes as $r_{m-1}$ and $r_\ell$ for $\ell < m - 1$.

In the former case, $r_n$ cooks another dish; but every dish besides $d_1$ is already cooked by two chefs in $T_n$, so $r_n$ must also cook $d_1$, closing the cycle. Because every resident is in this cycle, the statement to be proven is also true for $T_n$.

In the latter case, the same logic shows that $r_m$ cooks $d_1$, also closing the cycle, but there are other residents of $T_n$ who have yet to be accounted for. Let $C_1 = \{r_1, \ldots, r_m\}$, and consider the town $T'$ whose residents are $T_n \setminus C_1$. Each of dishes $d_1, \ldots, d_m$ is cooked by two people in $C_1$, so no chef in $T'$ cooks any of these dishes, and no dish in $T'$ is cooked by any of the people in $C_1$ (because each person in $C_1$ already cooks two dishes in the set $\text{dish}(C_1)$). Thus $|\text{chef}_{T'}(D)| = 2$ for each $D$ in $\text{dish}(T')$. It follows that $\text{pop}(T') < \text{pop}(T)$ but $\text{pop}(T') > 0$, so by the inductive hypothesis, the residents of $T'$ can be divided into disjoint resident cycles.

Thus the statement is proved by strong induction.

**b.** In order for $T$ to be homogeneous, it must be possible to partition the residents into exactly two dining groups. First apply 10a to divide the town into finitely many resident cycles $C_i$, and assume towards a contradiction that such a group assignment $f : T \to \{1, 2\}$ exists. If $\text{pop}(T)$ is odd, then at least one of the cycles $C_i$ must contain an odd number of residents; without loss of generality, suppose this cycle to be $C_1$, with residents $r_1, r_2, \ldots, r_{2k+1}$. (By the restrictions noted in part a, $k \geq 1$.) Now because $r_i$ and $r_{i+1}$ cook a dish in common, $f(r_i) \neq f(r_{i+1})$ for all $i$. Thus if $f(r_1) = 1$, it follows that $f(r_2) = 2$, and that $f(r_3) = 1$, etc. So $f(r_i) = f(r_1)$ if $i$ is odd and $f(r_i) = f(r_2)$ if $i$ is even; in particular, $f(r_{2k+1}) = f(1)$. But that equation would imply that $r_1$ and $r_{2k+1}$ cook no dishes in common, which is impossible if they are the first and last residents in a resident cycle. So no such group assignment can exist, and $\text{gr}(T) \geq 3$. Hence $T$ is heterogeneous.

**11. a.** In problem 5, it was shown that

$$2\,\text{pop}(T) = \sum_{D \in \text{dish}(T)} |\text{chef}_T(D)|\,.$$

Therefore $\sum_{D \in \text{dish}(T)} |\text{chef}_T(D)|$ is even. But if $|\text{chef}_T(D)| = 3$ for all $D \in \text{dish}(T)$, then the sum is simply $3\,|\text{dish}(T)|$, so $|\text{dish}(T)|$ must be even.

**b.** By problem 6, it must be the case that $\text{gr}(T) \geq 3$. Let $C = \{r_1, r_2, \ldots, r_n\}$ denote a resident cycle such that for every dish $D \in \text{dish}(T)$, there exists a chef in $C$ that can prepare $D$. Each resident is a chef for two dishes, and every dish can be made by two residents in $C$ (although by three in $T$). Thus the number of residents in the resident cycle $C$ is equal to $|\text{dish}(T)|$, which was proved to be even in the previous part.

Define a group assignment by setting

$$f(r) = \begin{cases} 1 & \text{if } r \notin C \\ 2 & \text{if } r = r_i, i \text{ is even} \\ 3 & \text{if } r = r_i, i \text{ is odd.} \end{cases}$$

For any $D \in \text{dish}(T)$, there are exactly three $D$-chefs, and exactly two of them belong to the resident cycle $C$. Hence exactly one of the $D$-chefs $r$ will have $f(r) = 1$. The remaining two $D$-chefs will be $r_i$ and $r_{i+1}$ for some $i$, or $r_1$ and $r_n$. In either case, the group assignment $f$ will assign one of them to 2 and the other to 3. Thus any two residents who make a common dish will be assigned different groups by $f$, so $f$ is a valid group assignment, proving that $\text{gr}(T) = 3$.

**12.** **a.** From problem 5,

$$2\,\text{pop}(T) = \sum_{D \in \text{dish}(T)} |\text{chef}_T(D)|\,.$$

Because $|\text{chef}_T(D)| = k$ for all $D \in \text{dish}(T)$, the sum is $k \cdot \text{dish}(T)$. Thus $2\,\text{pop}\,T = k \cdot \text{dish}(T)$, and so $k \cdot \text{dish}(T)$ must be even. By assumption, $|\text{dish}(T)|$ is odd, so $k$ must be even.

**b.** Suppose for the sake of contradiction that there is some $n$ for which the group $R = \{r \in T \mid f(r) = n\}$ has a $D$-chef for every dish $D$. Because $f$ is a group assignment and $f$ assigns every resident of $R$ to group $n$, no two residents of $R$ make the same dish. Thus for every $D \in \text{dish}(T)$, exactly one resident of $R$ is a $D$-chef; and each $D$-chef cooks exactly one other dish, which itself is not cooked by anyone else in $R$. Thus the dishes come in pairs: for each dish $D$, there is another dish $D'$ cooked by the $D$-chef in $R$ and no one else in $R$. However, if the dishes can be paired off, there must be an even number of dishes, contradicting the assumption that $|\text{dish}(T)|$ is odd. Thus for every $n$, the set $\{r \in T \mid f(r) = n\}$ must be missing a $D$-chef for some dish $D$.

**c.** Let $f$ be a group assignment for $T$, and let $R = \{r \in T \mid f(r) = 1\}$. From problem 12b, there must be some $D \in \text{dish}(T)$ with no $D$-chefs in $R$. Moreover, $f$ cannot assign two $D$-chefs to the same group, so there must be at least $k$ other groups besides $R$. Hence there are at least $1 + k$ different groups, so $\text{gr}(T) > k$.

**13.** **a.** Fix $D \in \text{dish}(\mathcal{F}_d)$. Then for every other dish $D' \in \text{dish}(\mathcal{F}_d)$, there is exactly one chef who makes both $D$ and $D'$, hence $|\text{chef}_{\mathcal{F}_d}(D)| = d-1$, which is even because $d$ is odd. Thus for each $D \in \text{dish}(\mathcal{F}_d)$, $|\text{chef}_{\mathcal{F}_d}(D)|$ is even. Because $|\text{dish}(\mathcal{F}_d)| = d$ is odd and $|\text{chef}_{\mathcal{F}_d}(D)| = d-1$ for every dish in $\mathcal{F}_d$, problem 12c applies, hence $\text{gr}(\mathcal{F}_d) > d-1$.

Label the dishes $D_1, D_2, \ldots, D_d$, and label the residents $r_{i,j}$ for $1 \le i < j \le d$ so that $r_{i,j}$ is a $D_i$-chef and a $D_j$-chef. Define $f : \mathcal{F}_d \to \{0, 1, \ldots, d-1\}$ by letting $f(r_{i,j}) \equiv i + j \bmod d$.

Suppose that $f(r_{i,j}) = f(r_{k,\ell})$, so $i + j \equiv k + \ell \bmod d$. Then $r_{i,j}$ and $r_{k,\ell}$ are assigned to the same group, which is a problem if they are different residents but are chefs for the same dish. This overlap occurs if and only if one of $i$ and $j$ is equal to one of $k$ and $\ell$. If $i = k$, then $j \equiv \ell \bmod d$. As $j$ and $\ell$ are both between 1 and $d$, the only way they could be congruent modulo $d$ is if they were in fact equal. That is, $r_{i,j}$ is the same resident as $r_{k,\ell}$. The other three cases ($i = \ell$, $j = k$, and $j = \ell$) are analogous. Thus $f$ is a valid group assignment, proving that $\text{gr}(\mathcal{F}_d) \le d$. Therefore $\text{gr}(\mathcal{F}_d) = d$.

**b.** In problem 4, it was shown that the sequence $\mathrm{gr}(\mathcal{F}_2), \mathrm{gr}(\mathcal{F}_3), \ldots$ is nondecreasing. If $d$ is even, $\mathrm{gr}(\mathcal{F}_d) \geq \mathrm{gr}(\mathcal{F}_{d-1})$, and because $d-1$ is odd, problem 13a applies: $\mathrm{gr}(\mathcal{F}_{d-1}) = d-1$. Hence $\mathrm{gr}(\mathcal{F}_d) \geq d-1$. Now it suffices to show that $\mathrm{gr}(\mathcal{F}_d) \leq d-1$ by exhibiting a valid group assignment $f : \mathcal{F}_d \to \{1, 2, \ldots, d-1\}$.

Label the dishes $D_1, \ldots, D_d$, and label the residents $r_{i,j}$ for $1 \leq i < j \leq d$ so that $r_{i,j}$ is a $D_i$-chef and a $D_j$-chef. Let $R = \{r_{i,j} \mid i, j \neq d\}$. That is, $R$ is the set of residents who are not $D_d$-chefs. Using 1c, $R$ is a full town with $d-1$ dishes, so from 12a, it has a group assignment $f : R \to \{1, 2, \ldots, d-1\}$. For each $D_i \in \mathrm{dish}(\mathcal{F}_d)$, $i \neq d$, $|\,\mathrm{chef}_R(D_i)| = d-2$. Because there are $d-1$ groups and $|\mathrm{chef}_R(D_i)| = d-2$, exactly one group $n_i$ must not contain a $D_i$-chef for each dish $D_i$.

It cannot be the case that $n_i = n_j$ for $i \neq j$. Indeed, suppose for the sake of contradiction that $n_i = n_j$. Without loss of generality, assume that $n_i = n_j = 1$ (by perhaps relabeling the dishes). Then any resident $r \in R$ assigned to group 1 (that is, $f(r) = 1$) would be neither a $D_i$-chef nor a $D_j$-chef. The residents in $R$ who are assigned to group 1 must all be chefs for the remaining $d-3$ dishes. Because each resident cooks two dishes, and no two residents of group 1 can make a common dish,

$$|\{r \in R \mid f(r) = 1\}| \leq \frac{d-3}{2}.$$

For each of the other groups $2, 3, \ldots, d-1$, the number of residents of $R$ in that group is no more than $(d-1)/2$, because there are $d-1$ dishes in $R$, each resident cooks two dishes, and no two residents in the same group can make a common dish. However, because $d-1$ is odd, the size of any group is actually no more than $(d-2)/2$. Therefore

$$\begin{aligned}
|R| &= \sum_{k=1}^{d-1} |\{r \in R \mid f(r) = k\}| \\
&= |\{r \in R \mid f(r) = 1\}| + \sum_{k=2}^{d-1} |\{r \in R \mid f(r) = k\}| \\
&\leq \frac{d-3}{2} + \sum_{k=2}^{d-1} \frac{d-2}{2} \\
&= \frac{d-3}{2} + \frac{(d-2)^2}{2} \\
&= \frac{d^2 - 3d + 1}{2} < \frac{d^2 - 3d + 2}{2} = |R|.
\end{aligned}$$

This is a contradiction, so it must be that $n_i \neq n_j$ for all $i \neq j$, making $f$ a valid group assignment on $\mathcal{F}_d$. Hence $\mathrm{gr}(\mathcal{F}_d) = d-1$.