CONNECTICUT STATE ASSOCIATION OF MATH LEAGUES
State Math Match, 2023

Please write your answers on the answer sheet provided.

Round 1: Arithmetic and Number Theory

1-1 The price of an object is initially \$80. The price is increased by 15%, and then by 15% again. Find the final price of the object in cents. (Do not include a unit in your answer.)

1-2 A cashier needs to give a customer 23 cents in change. The change is to be made with k coins consisting of pennies, nickels, and/or dimes. Find the number of possible values of k.
(Note: A penny is worth 1 cent, a nickel is worth 5 cents, and a dime is worth 10 cents.)

1-3 The letters A, B, and C represent different digits. A is prime, $A - B = 4$, and the number $\underline{A}\,\underline{A}\,\underline{A}\,\underline{B}\,\underline{B}\,\underline{B}\,\underline{C}$ is prime. Find $A + 2B + 3C$.
(Note: The notation $\underline{R}\,\underline{S}\,\underline{T}$ represents the number whose digits are R, S, T, in that order.)

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Round 2: Algebra I

2-1   If $27^{4x} = \left(9\sqrt{3}\right)^{2x+1}$, then $x = \frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

2-2   If one of the roots of the equation $x^2 + 8x + p = 0$ is 5 times the other, then $p = \frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

2-3   Let $Q(x)$ be the quotient when $23x^{100} + x^{20} + 5$ is divided by $(x - 1)$. Find the sum of the coefficients of $Q(x)$.

Notes:
- Example of the meaning of "quotient": When 7 is divided by 3, the quotient is 2.
- The sum of the coefficients of the polynomial $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ is $a_n + a_{n-1} + \cdots + a_0$.

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Round 3: Geometry

3-1   Find the surface area of a cube that has a diagonal of length 6.
       (Note: A diagonal of a cube is a line segment joining a vertex of the cube to the furthest
       other vertex of the cube.)

3-2   Let $ABCDE$ be a regular pentagon and let F be that point in the interior of the pentagon
       such that ABF is an equilateral triangle. Find the degree measure of angle FED.
       (Do not include a unit in your answer.)

3-3   Semicircles are drawn on two sides of the square $ABCD$, as shown in the diagram
       below. Point E is the center of the square, and $\overline{QAP}$ is a line segment with $QA = 17$
       and $AP = 31$. Find the distance AE.
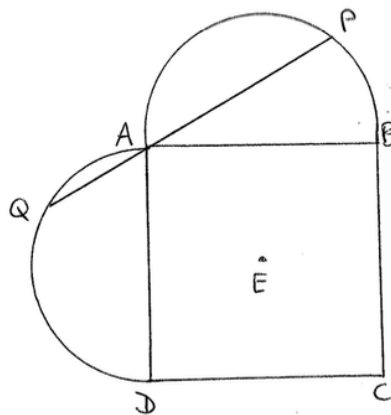
Please write your answers on the answer sheet provided.

Round 4: Algebra II

4-1   Let $f(x) = x^2 + 2x$ and $g(x) = 2x - 1$. Find $f\left(g^{-1}(5)\right)$.

4-2   Find the sum of all solutions of the equation

$$\log_6(x - 3) + \log_6(x - 2) = 1 + \log_6 5$$

4-3   The rows of Pascal's triangle (shown below) are written successively in order to form the sequence 1, 1, 1, 1, 2, 1, 1, 3, 3, 1, 1, 4, 6, 4, 1, 1, 5, 10, 10, … . The sum of the first 213 terms of the sequence is $2^p + q$, where p and q are positive integers and $q < 2^p$. Find $p + q$.

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
etc.

Please write your answers on the answer sheet provided.

Round 5: Analytic Geometry

5-1   A parabola has equation $y = x^2 + Bx + C$, where B and C are constants. The parabola passes through the points $(-2, 5)$ and $(5, 12)$. Find $B^2 + C^2$.

5-2   An ellipse has equation $2x^2 + 3y^2 - 12x + 12y - 6 = 0$. The product of the lengths of the major and minor axes of this ellipse is $a\sqrt{b}$, where a and b are positive integers and b is not divisible by the square of any prime number. Find $a + b$.

5-3   (Note: The tangent line to the curve $y = x^2$ at the point (a, a^2) has slope $2a$.)
Suppose that the angle formed by the tangents to the curve $y = x^2$ from the point (r, s) in Quadrant II is bisected by the line through (r, s) with slope 1. Then $s = \frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

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Round 6: Trigonometry and Complex Numbers

6-1   For how many integers n with $1 \le n \le 200$ is $\sin n° > \frac{1}{2}$?
       (Note: " $n°$ " means "n degrees".)

6-2   Let $z = 3\sqrt{3} + 3i$ and $w = \cos 15° + i \sin 15°$. Find the absolute value of the real part
       of the complex number $\sqrt{2} \cdot z^4 \cdot w$.

6-3   Find the smallest positive value of x, <u>in degrees</u>, for which the function

$$f(x) = \sin \frac{x}{3} + \sin \frac{x}{11}$$

       achieves its maximum value. (Do not include a unit in your answer.)

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Team Round

T-1   How many multiples of 23 are there between 1 and 1,000,000 that are even and perfect squares?

T-2   Suppose that $\frac{x-y+z}{x+y+2z} = \frac{1}{4}$ and $\frac{x+y}{x+y+z} = \frac{1}{2}$. Then the ratio $x:y:z$ is $a:b:c$, where a, b, c are positive integers and $\gcd(a,b,c) = 1$. Find $a + b + c$.

T-3   An isosceles trapezoid has bases whose lengths are a and b, where $a > b$. Suppose that the height of the trapezoid is b and the length of a diagonal of the trapezoid is a. Then $\frac{a}{b} = \frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

T-4   Let $a_1, a_2, a_3, \ldots$ be a strictly increasing sequence of positive integers in which each term is equal to the sum of the previous two terms. If the numbers 305 and 2023 are both terms of the sequence, what is the least possible value of a_1 ?

T-5   Circle C_1 has equation $x^2 + y^2 - 8x - 6y - 11 = 0$ and circle C_2 has equation $(x - 9)^2 + (y - a)^2 = 49$, where a is a constant and $a > 0$. If C_1 and C_2 are externally tangent, what is the value of a ?

T-6   Triangle ABC is drawn on the coordinate plane such that B is at the origin, C is located at the point $(4, 2\sqrt{5})$, and A is located in Quadrant II such that $AB = 8$ and $AC = 2\sqrt{33}$. The x-coordinate of point A is $-\frac{a+b\sqrt{c}}{d}$, where $a, b, c,$ and d are positive integers, c is not divisible by the square of any prime number, and $\gcd(a,b,d) = 1$. Find $a + b + c + d$.

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Answers

Round 1		Team Round	
1-1	10580	T-1	21
1-2	9	T-2	16
1-3	16	T-3	8
		T-4	27
		T-5	15
Round 2		T-6	51

Round 2	
2-1	19
2-2	89
2-3	2320

Round 3

3-1	72
3-2	42
3-3	25

Round 4

4-1	15
4-2	8
4-3	230

Round 5

5-1	13
5-2	30
5-3	5

Round 6

6-1	119
6-2	1296
6-3	8910