CONNECTICUT STATE ASSOCIATION OF MATH LEAGUES
State Math Match, 2023
Solutions

Round 1: Arithmetic and Number Theory

1-1 The price of an object is initially \$80. The price is increased by 15%, and then by 15% again. Find the final price of the object in cents. (Do not include a unit in your answer.)

Solution:
Increasing the price by 15% is equivalent to multiplying by 1.15. So, \$80 increased by 15% and then 15% again can be calculated as follows: $80(1.15)(1.15) = 105.8$
So, the price after increase is \$105.80, and this value converted to cents is 10,580.

1-2 A cashier needs to give a customer 23 cents in change. The change is to be made with k coins consisting of pennies, nickels, and/or dimes. Find the number of possible values of k.
(Note: A penny is worth 1 cent, a nickel is worth 5 cents, and a dime is worth 10 cents.)

Solution:
Let $p, n,$ and d represent the number of pennies, nickels, and dimes respectively. We then know the following equations must be true:

$$p + 5n + 10d = 23 \text{ and } p + n + d = k.$$

Notice that the expression $5n + 10d$ can produce only multiples of 5. So $23 - p$ is a non-negative multiple of 5. In particular, $p \geq 3$ which means that in any case $k > 3$ (since we need at least 3 pennies, but 3 pennies alone is not enough). Subtracting the second equation above from the first yields:

$$4n + 9d = 23 - k$$
$$k = 23 - (4n + 9d)$$

Since k must be a value greater than 3, and n and d are non-negative, the expression $4n + 9d$ must be at least 0, but at most 19. So, we need to check which values between 0 and 19 (inclusive) the expression $4n + 9d$ can take on when using non-negative values for each variable. We see that with these constraints, the expression $4n + 9d$ can take on the following values: 0, 4, 8, 9, 12, 13, 16, 17, and 18. Each of these produces a new value for k, meaning that there are 9 different possible values for k.

1-3 The letters A, B, and C represent different digits. A is prime, $A - B = 4$, and the number $\underline{A}\,\underline{A}\,\underline{A}\,\underline{B}\,\underline{B}\,\underline{B}\,\underline{C}$ is prime. Find $A + 2B + 3C$.
(Note: The notation $\underline{R}\,\underline{S}\,\underline{T}$ represents the number whose digits are R, S, T, in that order.)

Solution:
For convenience, let $\underline{A}\,\underline{A}\,\underline{A}\,\underline{B}\,\underline{B}\,\underline{B}\,\underline{C} = n$. Since $A - B = 4$ and A is prime, the possibilities for A and B are $A = 5$ and $B = 1$, or $A = 7$ and $B = 3$.

Note that, for n to be prime, C must be odd and not 5.

Suppose that $A = 5$ and $B = 1$. Then the possibilities for C are 3, 7, 9. However, if $C = 3$ or $C = 9$, n would be divisible by 3. So, we are left with $C = 7$, making $n = 5551117$. However, this number is divisible by 11. (Use the rule for divisibility by 11, which is easily found online.)

Now suppose that $A = 7$ and $B = 3$. Then the possibilities for C are 1 and 9. However, if $C = 9$ then n is divisible by 3. Hence, $C = 1$.

So, $A + 2B + 3C = 7 + 2 \cdot 3 + 3 \cdot 1 = 16$.

Round 2: Algebra I

2-1 If $27^{4x} = \left(9\sqrt{3}\right)^{2x+1}$, then $x = \frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

Solution:
Note that $27 = 3^3$, $9 = 3^2$, and $\sqrt{3} = 3^{\frac{1}{2}}$. Making these substitutions, our equation becomes:

$$(3^3)^{4x} = \left(3^2 \cdot 3^{\frac{1}{2}}\right)^{2x+1}$$

Applying some properties of exponents:

$$3^{12x} = 3^{5x+\frac{5}{2}}$$

And since exponential functions are one-to-one:

$$12x = 5x + \frac{5}{2}$$
$$7x = \frac{5}{2}$$
$$x = \frac{5}{14}$$

Since 5 and 14 are relatively prime positive integers, we have $a + b = 5 + 14 = 19$.

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2-2 If one of the roots of the equation $x^2 + 8x + p = 0$ is 5 times the other, then $p = \frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

Solution:
Note that neither root can be zero since then we would have $p = 0$ and the other root would be −8, contradicting the constraint that one root is 5 times the other. Let the roots of the equation be r and $5r$. This means we have the following system:

$$\text{(eq1)} \quad r^2 + 8r + p = 0 \quad \text{and}$$
$$\text{(eq2)} \quad (5r)^2 + 8(5r) + p = 0$$

Simplifying equation 2, our new system is:

$$\text{(eq1)} \quad r^2 + 8r + p = 0 \quad \text{and}$$
$$\text{(eq2)} \quad 25r^2 + 40r + p = 0$$

We then subtract equation 1 from equation 2 and solve the resulting equation for r:

$$24r^2 + 32r = 0$$
$$8r(3r + 4) = 0$$

Since $r \neq 0$ we must have $3r + 4 = 0$, meaning $r = -\frac{4}{3}$. So, the other root is five times this, namely $-\frac{20}{3}$. With the two roots, we can produce a unique quadratic in standard form with a leading coefficient of 1 by starting with factored form and multiplying:

$$\left(x + \frac{4}{3}\right)\left(x + \frac{20}{3}\right) = 0$$

$$x^2 + 8x + \frac{80}{9} = 0$$

Comparing this with the given equation, we see that $p = \frac{80}{9}$ and since 80 and 9 are relatively prime, we have $a + b = 80 + 9 = 89$.

Alternative Solution
For some r, the roots of the equation are r and $5r$. Using Vieta's formulas, $r + 5r = -8$ and $r \cdot 5r = p$. Using the first equation, $r = -\frac{4}{3}$. Therefore, using the second equation, $p = 5r^2 = 5 \cdot \frac{16}{9} = \frac{80}{9}$. So, the answer to the question is $80 + 9 = 89$.

2-3 Let $Q(x)$ be the quotient when $23x^{100} + x^{20} + 5$ is divided by $(x - 1)$. Find the sum of the coefficients of $Q(x)$.

Notes:

- Example of the meaning of "quotient": When 7 is divided by 3, the quotient is 2.
- The sum of the coefficients of the polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is $a_n + a_{n-1} + \cdots + a_0$.

Solution:
Note, first, that
$23x^{100} + x^{20} + 5$
$= 23(x^{100} - 1) + (x^{20} - 1) + 29$
$= 23(x - 1)(x^{99} + x^{98} + \cdots + 1) + (x - 1)(x^{19} + x^{18} + \cdots + 1) + 29.$

So, $Q(x) = 23(x^{99} + \cdots + 1) + (x^{19} + \cdots + 1)$.

Note that the sum of the coefficients of $Q(x)$ is $Q(1)$. Thus, we need

$$Q(1) = 23 \cdot 100 + 20 = 2320.$$

Alternative Solution (Uses Calculus):
First note that $Q(x)$ is that polynomial such that

$$23x^{100} + x^{20} + 5 = (x - 1) \cdot Q(x) + R,$$

where R is a constant.

Differentiating both sides with respect to x we get

$$2300x^{99} + 20x^{19} = (x - 1) \cdot Q'(x) + Q(x).$$

Note again that the sum of the coefficients of $Q(x)$ is $Q(1)$. Substituting $x = 1$ in the equation above we get

$$2300 + 20 = 0 \cdot Q'(1) + Q(1).$$

Thus, the sum of the coefficients of $Q(x)$ is $Q(1) = 2320$.

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Round 3: Geometry

3-1 Find the surface area of a cube that has a diagonal of length 6.
(Note: A diagonal of a cube is a line segment joining a vertex of the cube to the furthest other vertex of the cube.)

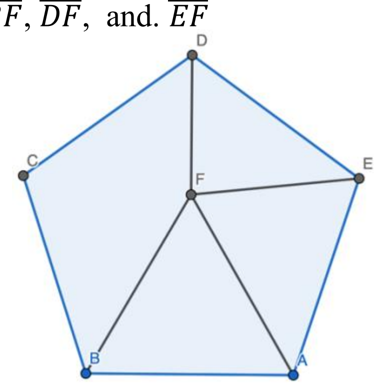
Solution:
The length of a diagonal of a cube can be found with the following formula: $d = s\sqrt{3}$ where d is the length of the diagonal and s is the side length of the cube. We can use this to solve for the side length:

$$s\sqrt{3} = 6$$
$$s = \frac{6}{\sqrt{3}}$$
$$s = 2\sqrt{3}$$

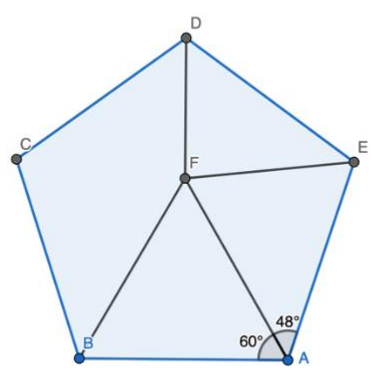
Now the formula for the surface area of a cube is $A = 6s^2$.
So, we have $A = 6\left(2\sqrt{3}\right)^2 = 6(12) = 72$

3-2 Let $ABCDE$ be a regular pentagon and let F be that point in the interior of the pentagon such that ABF is an equilateral triangle. Find the degree measure of angle FED.
(Do not include a unit in your answer.)

Solution:
Construct segments $\overline{AF}$, $\overline{BF}$, $\overline{DF}$, and. $\overline{EF}$

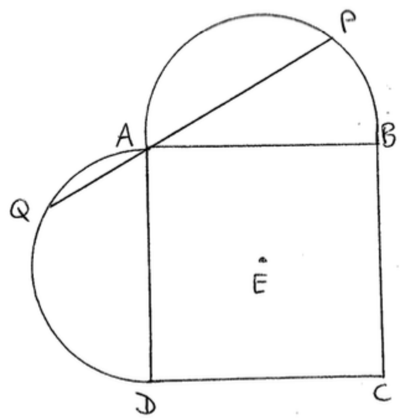


First note that in any regular pentagon, the measure of each interior angle is $108°$ and in an equilateral triangle, the measure of each interior angle is $60°$. These pieces of information together allow us to determine that $\mathrm{m}\angle FAE = 48°$.



We are given that triangle ABF is equilateral. So in particular, we can deduce that $\overline{AF} \cong \overline{AB}$. And since the pentagon is regular, we also know that $\overline{AB} \cong \overline{AE}$. This means that $\overline{AF} \cong \overline{AE}$, making triangle AFE isosceles. Since the base angles of any isosceles triangle are congruent we now know that $\text{m}\angle AEF = \text{m}\angle AFE = 66°$ because the sum of the interior angle measures of triangle AFE must be $180°$. Finally, since angle AED is an interior angle of the regular pentagon, its measure is $108°$ which means that $m\angle FED = 108° - 66° = 42°$.

3-3 Semicircles are drawn on two sides of the square $ABCD$, as shown in the diagram below. Point E is the center of the square, and $\overline{QAP}$ is a line segment with $QA = 17$ and $AP = 31$. Find the distance AE.



Solution:
Angles BPA and AQD are drawn in semicircles, and therefore are right angles. Note that $\text{m}\angle PBA = 90° - \text{m}\angle PAB = \text{m}\angle DAQ$. Also, the hypotenuses of triangles BPA and AQD are $\overline{AB}$ and $\overline{AD}$, respectively, which have equal lengths. Thus, triangles BPA and AQD are congruent. Therefore, $BP = 17$ and $DQ = 31$. Choose the point F on $\overline{QD}$ such that $\overline{BF} \perp \overline{QD}$. Then $BPQF$ is a rectangle and $BF = 31 + 17 = 48$, $QF = 17$. Therefore, $DF = 31 - 17 = 14$. Since triangle BFD is a right triangle, $DB^2 = BF^2 + FD^2 = 48^2 + 14^2 = 50^2$. Thus, $AE = \frac{1}{2}DB = \frac{1}{2} \cdot 50 = 25$.

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Round 4: Algebra II

4-1 Let $f(x) = x^2 + 2x$ and $g(x) = 2x - 1$. Find $f\left(g^{-1}(5)\right)$.

Solution:
We can calculate $g^{-1}(x)$ by letting $y = g(x)$, swapping x and y in $y = 2x - 1$ to produce a new equation, and solving for the new y in this new equation:

$$x = 2y - 1$$
$$y = \frac{x+1}{2}$$

So, we have $g^{-1}(x) = \frac{x+1}{2}$. Now we compose f and g^{-1} evaluated at 5:

$$f\left(g^{-1}(5)\right) = f\left(\frac{5+1}{2}\right) = f(3) = 3^2 + 2(3) = 15.$$

4-2 Find the sum of all solutions of the equation

$$\log_6(x-3) + \log_6(x-2) = 1 + \log_6 5$$

Solution:
First we will rearrange the equation so all of the logarithms are together on one side:

$$\log_6(x-3) + \log_6(x-2) - \log_6 5 = 1$$

Next we will use properties of logarithms to condense the three logarithms on the left into one:

$$\log_6\left(\frac{(x-3)(x-2)}{5}\right) = 1$$

Converting to exponential form yields:

$$\frac{(x-3)(x-2)}{5} = 6$$

Now we solve this quadratic equation:

$$(x-3)(x-2) = 30$$
$$x^2 - 5x + 6 = 30$$
$$x^2 - 5x - 24 = 0$$
$$(x-8)(x+3) = 0$$
$$x = 8 \text{ or } x = -3$$

However, since our original equation was $\log_6(x-3)+\log_6(x-2)=1+\log_6 5$ and logarithms are undefined for negative inputs, $x=-3$ is an extraneous solution. Therefore, the only solution (and therefore the sum of all solutions) is 8.

4-3 The rows of Pascal's triangle (shown below) are written successively in order to form the sequence 1, 1, 1, 1, 2, 1, 1, 3, 3, 1, 1, 4, 6, 4, 1, 1, 5, 10, 10, … . The sum of the first 213 terms of the sequence is 2^p+q, where *p* and *q* are positive integers and $q<2^p$. Find $p+q$.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
etc.

<u>Solution:</u>
Refer to the initial row of Pascal's triangle ("1") as the 0th row, "1 1" as the 1st row, "1 2 1" as the 2nd row, and so on. Note that, for any positive integer *k*, the number of numbers in the *k*th row of Pascal's triangle is $k+1$.

Suppose that the 213th number in the given sequence occurs in the $(n+1)$th row of Pascal's triangle. Then $1+2+\cdots+(n+1)<213$ and $1+2+\cdots+(n+2)\geq 213$. So $\frac{1}{2}(n+1)(n+2)<213$ and $\frac{1}{2}(n+2)(n+3)\geq 213$. Guessing and checking, we find that $n=19$. So, the 213th term of the sequence occurs in the 20th row of Pascal's triangle.

The number of terms of the sequence accounted for by the 0th through 19th rows of Pascal's triangle is $1+2+\cdots+20=\frac{1}{2}\cdot 20\cdot 21=210$. Thus, the 213th term of the sequence is the 3rd number in the 20th row of Pascal's triangle. The first three numbers in row 20 are $\binom{20}{0},\binom{20}{1},\binom{20}{2}$; that is, 1, 20, 190.

It is known that, for any positive integer *k*, the sum of the numbers in the *k*th row of Pascal's triangle is 2^k. (Think of the expansion of $(1+1)^k$.) So, the sum of the first 213 terms of the sequence is $2^0+2^1+\cdots+2^{19}+1+20+190=\frac{1(1-2^{20})}{1-2}+211$ $=(20^{20}-1)+211=2^{20}+210$. (Note that $210<2^{20}$.) Thus, $p=20$ and $q=210$, making $p+q=230$.

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Round 5: Analytic Geometry

5-1 A parabola has equation $y = x^2 + Bx + C$, where B and C are constants. The parabola passes through the points $(-2, 5)$ and $(5, 12)$. Find $B^2 + C^2$.

Solution:
Any point the parabola passes through will satisfy the equation and hence make the equation true. Therefore, we can substitute the pair $x = -2$ and $y = 5$ as well as the pair $x = 5$ and $y = 12$ into the given equation to write the following linear system in B and C:

$$5 = (-2)^2 + B(-2) + C$$
$$12 = 5^2 + B(5) + C$$

Simplifying and rearranging some terms we get the system:

$$-2B + C = 1$$
$$5B + C = -13$$

Subtracting the first from the second we have:

$$7B = -14$$

And therefore $B = -2$. We can now substitute this into $-2B + C = 1$ and solve for C:

$$-2(-2) + C = 1$$
$$C = -3$$

So, we have $B^2 + C^2 = (-2)^2 + (-3)^2 = 4 + 9 = 13$

5-2 An ellipse has equation $2x^2 + 3y^2 - 12x + 12y - 6 = 0$. The product of the lengths of the major and minor axes of this ellipse is $a\sqrt{b}$, where a and b are positive integers and b is not divisible by the square of any prime number. Find $a + b$.

Solution:

The standard form for the equation of an ellipse is: $\frac{(x-h)^2}{A^2}+\frac{(y-k)^2}{B^2}=1$ where $2A$ and $2B$ give the lengths of the major and minor axes (note that A and B here are different from the a and b being defined by this question). We can write the equation of this ellipse in standard form by completing the square for the x terms and the y terms:

$$2x^2-12x=2(x^2-6x)=2((x-3)^2-9)=2(x-3)^2-18$$
$$3y^2+12y=3(y^2+4y)=3((y+2)^2-4)=3(y+2)^2-12$$

We can now substitute these into the original equation given:

$$2(x-3)^2-18+3(y+2)^2-12-6=0$$

Simplifying and rearranging terms:

$$2(x-3)^2+3(y+2)^2=36$$

Divide both sides by 36 to put it in standard form:

$$\frac{(x-3)^2}{18}+\frac{(y+2)^2}{12}=1$$

So we have $A^2=18$ and $B^2=12$, which implies $A=3\sqrt{2}$ and $B=2\sqrt{3}$. The product of the major and minor axes is then $2A\cdot 2B=\left(6\sqrt{2}\right)\left(4\sqrt{3}\right)=24\sqrt{6}$. So the a and b values defined by the problem are 24 and 6 and we have $a+b=24+6=30$

5-3 (Note: The tangent line to the curve $y=x^2$ at the point (a,a^2) has slope $2a$.) Suppose that the angle formed by the tangents to the curve $y=x^2$ from the point (r,s) in Quadrant II is bisected by the line through (r,s) with slope 1. Then $s=\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Solution:

Suppose that the points of tangency of the two tangent lines are (a,a^2) and (b,b^2). Then, the first of the two tangents has equation $y-a^2=2a(x-a)$, which simplifies to $y=2ax-a^2$. Similarly, the other tangent has equation $y=2bx-b^2$. Eliminating x from these two equations we get $(b-a)y=ab^2-a^2b$, which tells us that at the point of intersection of the two tangents, $y=ab$. That is, $s=ab$.

We are told that the angle formed by the two tangents is bisected by a line of slope 1. It follows from this that the slopes of the tangents are reciprocals of each other; that is, their product is 1. Hence $(2a)(2b)=1$, telling us that $ab=\frac{1}{4}$. Hence, $s=\frac{1}{4}$, and the answer to the question is 5.

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Round 6: Trigonometry and Complex Numbers

6-1 For how many integers n with $1 \leq n \leq 200$ is $\sin n° > \frac{1}{2}$?
(Note: " $n°$ " means "n degrees".)

Solution:
Consider the function $\sin x$, with x measured in degrees. This function is equal to 0 when $x = 0$, is equal to 1 when $x = 90$, and is strictly increasing on the interval from 0 to 90. It is exactly equal to $\frac{1}{2}$ when $x = 30$. Since it is strictly increasing from 0 to 90, it does not reach $\frac{1}{2}$ at any other point in this interval. From 90 to 270 the function is strictly decreasing from a value of 1 at $x = 90$ to a value of -1 at $x = 270$. Since it is continuous and strictly decreasing on this interval, it attains the value of $\frac{1}{2}$ exactly once on this interval, namely when $x = 150$. It does not reach $\frac{1}{2}$ again until $x = 390$, which is outside the given range of the problem. So we only need to count the integers between 31 and 149 inclusive (we exclude 30 and 150 because $\frac{1}{2}$ is not greater than $\frac{1}{2}$). There are 119 integers between 31 and 149 inclusive, all of which have a sine value greater than $\frac{1}{2}$.

6-2 Let $z = 3\sqrt{3} + 3i$ and $w = \cos 15° + i \sin 15°$. Find the absolute value of the real part of the complex number $\sqrt{2} \cdot z^4 \cdot w$.

Solution:
$z = 6\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = 6(\cos 30° + i \sin 30°)$.
So, $z^4 = 6^4(\cos(4 \cdot 30°) + i \sin(4 \cdot 30°)) = 6^4(\cos 120° + i \sin 120°)$.
Thus, $\sqrt{2} \cdot z^4 \cdot w = \sqrt{2} \cdot 6^4(\cos(120° + 15°) + i \sin(120° + 15°))$
$= \sqrt{2} \cdot 6^4(\cos 135° + i \sin 135°) = \sqrt{2} \cdot 6^4\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$.
So, the absolute value of the real part of $\sqrt{2} \cdot z^4 \cdot w$ is $\left|\sqrt{2} \cdot 6^4 \cdot -\frac{\sqrt{2}}{2}\right| = 6^4 = 1296$.

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6-3 Find the smallest positive value of x, in degrees, for which the function

$$f(x) = \sin\frac{x}{3} + \sin\frac{x}{11}$$

achieves its maximum value. (Do not include a unit in your answer.)

Solution:
Note that the maximum value of $\sin\theta$ is 1, and that this value is attained when (and only when) $\theta = (4k+1)\cdot 90°$ for some integer k. Thus, the maximum possible value of f is 2, occurring when $\frac{x}{3} = (4m+1)\cdot 90°$ and $\frac{x}{11} = (4n+1)\cdot 90°$, for integers m and n. Additionally, we are looking for a positive value of x, so $m, n \geq 0$.

It follows that $x = 3(4m+1)\cdot 90°$ and $x = 11(4n+1)\cdot 90°$, so $3(4m+1) = 11(4n+1)$. It follows from this that $11n - 3m = -2$. There are infinitely many solutions to this equation, and as n increases, m does also; we want the smallest positive value of x, so we are looking for the solution in which m and n are nonnegative and minimized. By trial and error, we find this solution to be $n = 2,\ m = 8$. Therefore, the required value of x is $3(4\cdot 8+1)\cdot 90$ $= 3\cdot 33\cdot 90 = 8910$.

Team Round

T-1 How many multiples of 23 are there between 1 and 1,000,000 that are even and perfect squares?

Solution:
For a multiple of 23 to be a perfect square, it must be a multiple of 23^2. For it to be an *even* perfect square, it must be a multiple of 4. So, for all integers n, the integer $4\cdot 23^2\cdot n^2$ is even, a perfect square, and a multiple of 23. Furthermore, *any* number that is even, a perfect square, and a multiple of 23 has this form. So, we need to determine the greatest value of n for which $4\cdot 23^2\cdot n^2 \leq 1{,}000{,}000$. Simplifying this inequality, we have:

$$23^2 n^2 \leq 250{,}000$$
$$23n \leq 500$$

Either long division of $\frac{500}{23}$ (only until the units place) or through trial and error, we see that since n is an integer, $n \leq 21$. Note that since the question asks about values starting from 1, we know that $n \geq 1$. Since there are 21 possible values for n, there are 21 even perfect square multiples of 23.

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T-2 Suppose that $\frac{x-y+z}{x+y+2z} = \frac{1}{4}$ and $\frac{x+y}{x+y+z} = \frac{1}{2}$. Then the ratio $x\!:y\!:z$ is $a\!:b\!:c$, where a, b, c are positive integers and $\gcd(a, b, c) = 1$. Find $a + b + c$.

Solution:
We can clear the fractions in both given equations (also called cross multiply) to obtain the following two equations:

$$\text{(eq 1)}\quad 4x - 4y + 4z = x + y + 2z$$
$$\text{(eq 2)}\quad 2x + 2y = x + y + z$$

Collecting like terms to one side, we have:

$$\text{(eq 1)}\quad 3x - 5y + 2z = 0$$
$$\text{(eq 2)}\quad x + y - z = 0$$

Now since we do not need to find particular solutions for x, y, and z, just the ratios we will introduce two new variables for convenience. Let $p = \frac{x}{y}$ and $q = \frac{z}{y}$. Now we can divide both equations by y so our new variables appear:

$$\text{(eq 1)}\quad 3\left(\frac{x}{y}\right) - 5 + 2\left(\frac{z}{y}\right) = 0$$
$$\text{(eq 2)}\quad \frac{x}{y} + 1 - \frac{z}{y} = 0$$

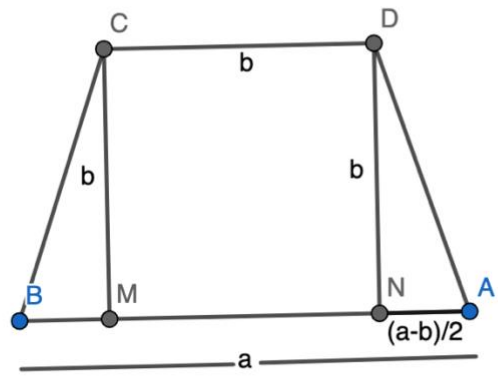
Now rewrite these equations using p and q:

$$\text{(eq 1)}\quad 3p - 5 + 2q = 0$$
$$\text{(eq 2)}\quad p + 1 - q = 0$$

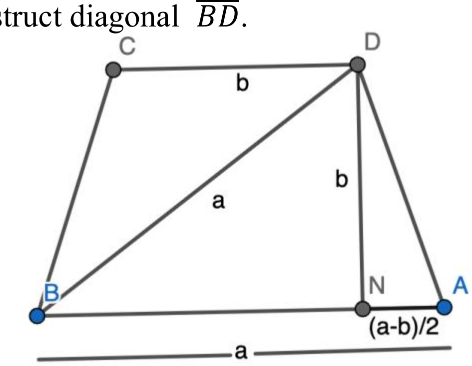
This is now a system of equations that we can solve to determine that $p = \frac{3}{5}$ and $q = \frac{8}{5}$. So $\frac{x}{y} = \frac{3}{5}$ and $\frac{z}{y} = \frac{8}{5}$. We can take the reciprocal of the second result to see that $\frac{y}{z} = \frac{5}{8}$. So the ratio $x\!:y\!:z$ is $3\!:5\!:8$, all of which are relatively prime. Therefore $a + b + c = 3 + 5 + 8 = 16$.

T-3 An isosceles trapezoid has bases whose lengths are a and b, where $a > b$. Suppose that the height of the trapezoid is b and the length of a diagonal of the trapezoid is a. Then $\frac{a}{b} = \frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

<u>Solution:</u>
The solution will refer to the following diagram of such a trapezoid:



In the diagram, the trapezoid is $ABCD$, point M is such that $\overline{CM} \perp \overline{AB}$, and point N is such that $\overline{DN} \perp \overline{AB}$. From the diagram, we can see that quadrilateral $CDNM$ is a square and so the length $MN = b$. And since the trapezoid is isosceles, triangle BMC is congruent to triangle AND. So in particular, $\overline{BM} \cong \overline{AN}$, which justifies the label in the diagram that $NA = \frac{a-b}{2}$. Now remove segment $\overline{CM}$ (as we only used this to justify the label on $\overline{NA}$) and construct diagonal $\overline{BD}$.



Now we can see that triangle BND is a right triangle and $BN = a - \frac{a-b}{2} = \frac{a+b}{2}$. Using the Pythagorean theorem, we can write the following equation:

$$\left(\frac{a+b}{2}\right)^2 + b^2 = a^2$$

$$\frac{1}{4}(a^2 + 2ab + b^2) + 4b^2 = a^2$$

$$a^2 + 2ab + 5b^2 = 4a^2$$

$$3a^2 - 2ab - 5b^2 = 0$$

Since we are looking for the ratio $\frac{a}{b}$ and not specific values, we will introduce a new variable for convenience. Let $u = \frac{a}{b}$. Now divide both sides of our equation by b^2 so our new variable appears:

$$3 \cdot \frac{a^2}{b^2} - 2 \cdot \frac{a}{b} - 5 = 0$$
$$3u^2 - 2u - 5 = 0$$

Solve this quadratic using any method you like to get $u = -1$ or $u = \frac{5}{3}$. Since our ratio must be a positive value, as it is a ratio of two lengths, the only valid solution is $u = \frac{5}{3}$, which means $\frac{a}{b} = \frac{5}{3}$, so we have $a + b = 5 + 3 = 8$.

T-4 Let $a_1, a_2, a_3, \ldots$ be a strictly increasing sequence of positive integers in which each term is equal to the sum of the previous two terms. If the numbers 305 and 2023 are both terms of the sequence, what is the least possible value of a_1 ?

Solution:
Let the number following 305 in the sequence be b. Then, from 305 onward the sequence is

$305, b, 305 + b, 305 + 2b, 2 \cdot 305 + 3b, 3 \cdot 305 + 5b, 5 \cdot 305 + 8b, 8 \cdot 305 + 13b, \ldots$.

Note that $8 \cdot 305 > 2023$, so one of the numbers $305 + b$, $305 + 2b$, $2 \cdot 305 + 3b$, $3 \cdot 305 + 5b$, $5 \cdot 305 + 8b$ must be 2023.

If $305 + b = 2023$ then $b = 1718$. So, 305 is followed by 1718 in the sequence; if there were a term before 305 in the sequence, that term would be equal to 1413, which is greater than 305 and therefore not allowed. Thus $a_1 = 305$. Working in a similar way, if $305 + 2b = 2023$ then, again, $a_1 = 305$.

Neither $3 \cdot 305 + 5b = 2023$ nor $5 \cdot 305 + 8b = 2023$ is possible, as in neither case would b be a whole number.

Now consider the case where $2 \cdot 305 + 3b = 2023$. Then $b = 471$. Tracking the sequence backward we get $471, 305, 166, 139, 27, 112$. Note that $112 > 27$, so 27 can't be preceded by 112. Thus, the smallest possible value of a_1 in this case is 27.

Note that $a_1 = 27$ is smaller than either of the possibilities of $a_1 = 305$ found above. Thus, the least possible value of a_1 is 27.

CONNECTICUT STATE ASSOCIATION OF MATH LEAGUES
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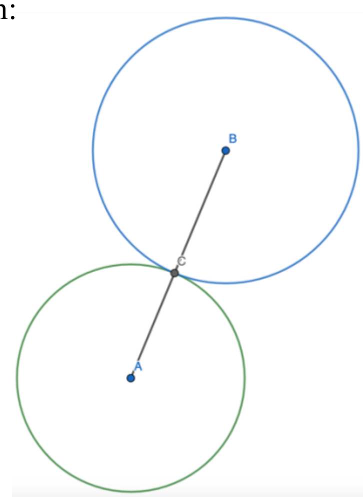
Solutions

T-5 Circle C_1 has equation $x^2 + y^2 - 8x - 6y - 11 = 0$ and circle C_2 has equation $(x-9)^2 + (y-a)^2 = 49$, where a is a constant and $a > 0$. If C_1 and C_2 are externally tangent, what is the value of a?

<u>Solution:</u>
The solution will refer to the following diagram:



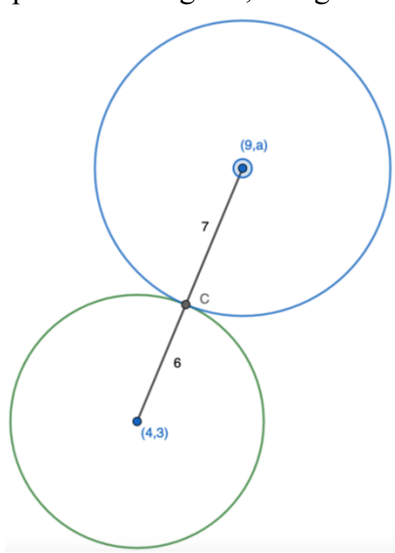
The circle centered at A is C_1 and the circle centered at B is C_2. We can complete the square for the x terms and for the y terms in the equation for circle one to write it in standard form:

$$x^2 - 8x = (x-4)^2 - 16$$
$$y^2 - 6y = (y-3)^2 - 9$$

So the equation of circle one becomes:

$$(x-4)^2 - 16 + (y-3)^2 - 9 - 11 = 0$$
$$(x-4)^2 + (y-3)^2 = 36$$

This means that circle one is centered at $(4,3)$ with radius 6. We can also see from the information given about the equation for circle 2 that its radius is 7 and the x-coordinate of its center is 9. We can update our diagram, using a to represent the unknown y-coordinate:



We can now see that the distance between points $(4,3)$ and $(9,a)$ must be equal to 13. So, we can use the distance formula to solve for a:

$$\sqrt{(9-4)^2+(a-3)^2}=13$$
$$\sqrt{25+(a-3)^2}=13$$
$$25+(a-3)^2=169$$
$$(a-3)^2=144$$
$$a-3=\pm 12$$
$$a=-9 \text{ or } a=15$$

Since we were told $a>0$, we must have $a=15$.

T-6 Triangle ABC is drawn on the coordinate plane such that B is at the origin, C is located at the point $(4, 2\sqrt{5})$, and A is located in Quadrant II such that $AB=8$ and $AC=2\sqrt{33}$. The x-coordinate of point A is $-\frac{a+b\sqrt{c}}{d}$, where a, b, c, and d are positive integers, c is not divisible by the square of any prime number, and $\gcd(a,b,d)=1$. Find $a+b+c+d$.

Solution:
Let x represent the x-coordinate of point A and y represent the y-coordinate of point A. So we know that the distance between (x,y) and $(0,0)$ is 8, and the distance between (x,y) and $\left(4,2\sqrt{5}\right)$ is $2\sqrt{33}$. We can model these two facts using the distance formula, resulting in the following equations:

$$\text{(eq 1)}\ \ x^2+y^2=64$$
$$\text{(eq 2)}\ \ (x-4)^2+\left(y-2\sqrt{5}\right)^2=132$$

Expanding the binomials in equation 2 results in:

$$\text{(eq 1)}\ \ x^2+y^2=64$$
$$\text{(eq 2)}\ \ x^2+y^2-8x-4\sqrt{5}\,y+36=132$$

Notice that the expression x^2+y^2 appears in equation 2 and that equation 1 asserts that the value of this expression is 64. With this in mind we can make a substitution:

$$\text{(eq 1)}\ \ x^2+y^2=64$$
$$\text{(eq 2)}\ \ 64-8x-4\sqrt{5}\,y+36=132$$

Next, we can simplify equation 2:

$$\text{(eq 1)}\ \ x^2 + y^2 = 64$$
$$\text{(eq 2)}\ \ -8x - 4\sqrt{5}\,y = 32$$

This results in a system of equations that we can solve by substitution. First solve for y in equation 2:

$$\text{(eq 1)}\ \ x^2 + y^2 = 64$$
$$\text{(eq 2)}\qquad y = \frac{2x+8}{-\sqrt{5}}$$

Substituting equation 2 into equation 1:

$$x^2 + \left(\frac{2x+8}{-\sqrt{5}}\right)^2 = 64$$
$$x^2 + \frac{(2x+8)^2}{5} = 64$$
$$5x^2 + (2x+8)^2 = 320$$
$$9x^2 + 32x + 64 = 320$$
$$9x^2 + 32x - 256 = 0$$

Solving this quadratic equation using the quadratic formula results in:

$$x = \frac{-16 \pm 16\sqrt{10}}{9}$$

Since we know the x-coordinate is supposed to be negative, we will take only the negative solution: $x = \frac{-16-16\sqrt{10}}{2} = -\frac{16+16\sqrt{10}}{2}$. Comparing this with the definitions of $a, b, c,$ and d stated in the problem, we can see that $a = b = 16$, $c = 10$, and $d = 9$. Therefore $a + b + c + d = 16 + 16 + 10 + 9 = 51$.