CONNECTICUT STATE ASSOCIATION OF MATH LEAGUES
State Math Match, 2024: Solutions

Round 1: Arithmetic and Number Theory

1-1 For any positive integer n, let $p(n)$ be the number of distinct prime numbers that divide n. For example, $p(12) = 2$, since the only prime numbers that divide 12 are 2 and 3. Find $p(2024^2)$.
[Answer: 3]

Solution: Notice that squaring the number does not introduce any new prime factors. So we only need the distinct prime factors of 2024, which are 2, 11, and 23. So the answer is 3.

1-2 Annabel is on an 8-day cycle in which she works 7 consecutive days, then gets 1 rest day. Boris is on a 6-day cycle in which he works 5 consecutive days, then gets 1 rest day. There are 366 days this year. If they start working on January 1st, how many rest days will they have together this year?
[Answer: 15]

Solution: We can start by thinking of the days of the year being numbered 1 – 366 as opposed to the typical numbering system. So Jan 1 is numbered 1, Jan 31 is numbered 31, Feb 1 is numbered 32 and so on. With this enumeration of the days of the year, we can notice that Annabel will have a rest day exactly on days that are multiples of 8. Boris will have a rest day on exactly the days that are multiples of 6. So they will have a rest day together on days that are multiples of both 8 and 6. A number is a multiple of 8 and 6 if and only if it is a multiple of 24. This means we need to count how many multiples of 24 there are in the integers from 1 to 366. To do this, we only need to divide 366 by 24 and ignore any remainder. When we do this, the quotient is 15.

1-3 Let $S = \{2^0, 2^1, 2^2, \dots, 2^{n-1}\}$, where $30 \le n \le 35$. Let T be a subset of S such that the sum of the elements of T is exactly $\frac{1}{5}$ of the sum of the elements of S. How many numbers are there in the set T?
[Answer: 16]

Solution
Expressed as a binary number, the sum of the elements of S is 111…111, where this number consists of n 1's. The sum of the elements of T is $\frac{1}{5}$ of this, and 5 in binary is 101.

Using long division, note that $1111 \div 101 = 11$, which we will write as 0011.
Furthermore, 11111111 (eight 1's) $\div\, 101 = 00110011$,
111111111111 (twelve 1's) $\div\, 101 = 001100110011$ and so on.

Note, also, that if n is <u>not</u> divisible by 4, then any number consisting of n 1's is <u>not</u> divisible by 101. So we see that n must be divisible by 4, so $n = 32$, and the sum of the elements of T, in binary, is $00110011 \ldots 0011$, where this number consists of 8 renditions of "0011".

Since this is the only way to express the sum of the elements of T as a binary number, we know that T must consist of the 1st, 2nd, 5th, 6th, 9th, 10th, … elements of S. Thus, the set T consists of 16 numbers.

<u>Round 2: Algebra I</u>

2-1 Find the area of the region bounded by the lines $y = 4 - x$, $y = 3x - 8$, and the y-axis.
[Answer: 18]

<u>Solution</u>:
The vertices of the triangle formed by the three lines are (0,4), (0, -8) and (3,1)
A= (1/2)(4-(-8))(3)=18

2-2 There are 34 houses in a street. Some of the houses have solar panels, the rest do not. If 10 of the houses that don't have solar panels now install solar panels, the number of houses that now have solar panels is equal to the number that originally did not. How many houses originally had solar panels?
[Answer:12]

<u>Solution</u>:
x= the number of houses that DO have solar panels originally
34-x = the number of houses that DO NOT have solar panels originally
x+10=34-x x=12

2-3 The sum of all <u>real</u> solutions of the equation

$$\frac{8x^2 + x + 9}{5x^2 + 8x + 4} = \frac{9x^2 + 2x + 10}{6x^2 + 9x + 5}$$

is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
[Answer: 10]

<u>Solution</u>

Let $f(x) = 8x^2 + x + 9$, $g(x) = 5x^2 + 8x + 8$, and $h(x) = x^2 + x + 1$. Then the given equation can be written as

$$\frac{f(x)}{g(x)} = \frac{f(x) + h(x)}{g(x) + h(x)}$$

(Note that the numerator and the denominator on both sides are nonzero for all real x, since the discriminant is negative in each case.)

Thus, the original equation is true if and only if

$$f(x)g(x) + f(x)h(x) = f(x)g(x) + g(x)h(x),$$

and this is true if and only if $h(x)\big(f(x) - g(x)\big) = 0$; that is,

$$h(x) \cdot (3x^2 - 7x + 1) = 0$$

Note that $h(x)$ is nonzero for all real x and the solutions of the equation $3x^2 - 7x + 1 = 0$ are real. So, the real solutions of the original equation are the solutions of the equation $3x^2 - 7x + 1 = 0$. Their sum is $\frac{7}{3}$, so the answer to the question is $7 + 3 = 10$.

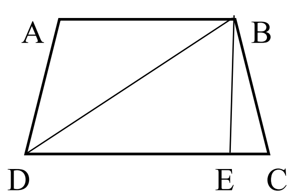
Round 3: Geometry

3-1 For a particular cube, the sum of the lengths of its edges in centimeters is numerically equal to its volume in cubic centimeters. Find the surface area of the cube in square centimeters. (Do not include a unit in your answer.)
[Answer: 72]

Solution:
$12x = x^3$, $x^2=12$, $6x^2=6(12)=72$

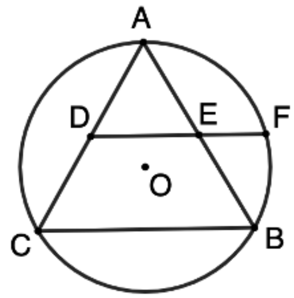
3-2 In an isosceles trapezoid, the length of each leg is 3, the length of each diagonal is 7, and the length of the shorter base is 5. Find the length of the longer base.
[Answer: 8]

Solution:



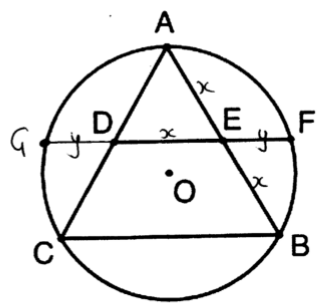
<DEB=90º DB=7, BC = 3 DE=5+x EC= x
$7^2-(5+x)^2=3^2-x^2$
$49 – 25 -10x-x^2 = 9 -x^2$
$x=3/2$ $5+ 2x = 8$

3-3 The diagram below shows an equilateral triangle ABC inscribed in a circle whose center is O. The points D and E are the midpoints of sides $\overline{AC}$ and $\overline{AB}$, respectively. Line $\overline{DE}$ intersects the circle at point F, with E between D and F. It follows that $\frac{DE}{EF} = \frac{a+\sqrt{b}}{c}$, where a, b, and c are positive integers and b is not divisible by the square of any prime number. Find $a + b + c$.



[Answer: 8]

<u>Solution</u>:
As shown in the diagram below, let G be the point where the line $\overline{ED}$ meets the circle, with $EF = DG = y$ and $DE = AE = EB = x$.



By the Power of a Point Theorem, $AE \cdot EB = GE \cdot EF$. So $x^2 = (y + x)y$, from which we get that $\left(\frac{x}{y}\right)^2 = 1 + \frac{x}{y}$. Using the quadratic formula, we find that $\frac{DE}{EF} = \frac{x}{y} = \frac{1 \pm \sqrt{5}}{2}$. Since the ratio must be positive, we have that $\frac{DE}{EF} = \frac{1+\sqrt{5}}{2}$, and the answer to the question is $1 + 5 + 2 = 8$.

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Round 4: Algebra II

4-1 Let $f(x) = x^2 - 2x + 11$ and $g(x) = \sqrt[3]{x+1}$. Find $f(g(7)) + g(f(-3))$.
[Answer: 14]

Solution:

First note $g(7) = \sqrt[3]{7+1} = 2$ and $f(-3) = (-3)^2 - 2(-3) + 11 = 26$, and then $f(g(7)) + g(f(-3)) = f(2) + g(26) = 2^2 - 2(2) + 11 + \sqrt[3]{26+1} = 11 + 3 = 14.$

4-2 An infinite geometric series has a positive common ratio. The sum of the first two terms is 5 and the sum of the entire series is 9. The common ratio can be written as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.
[Answer: 5]

Solution:

Let a be the first term and $r > 0$ represent the common ratio. From the description we have $a + ar = a(1 + r) = 5$, and $\frac{a}{1-r} = 9$. Therefore $a = \frac{5}{1+r}$, and $\frac{a}{1-r} = \left(\frac{5}{1+r}\right)\left(\frac{1}{1-r}\right) = \frac{5}{1-r^2} = 9$. Since $1 - r^2 = \frac{5}{9}$, we have $r^2 = \frac{4}{9}$ and therefore $r = \frac{2}{3}$, making the desired quantity $2 + 3 = 5$.

4-3 Let $P(x)$ be a polynomial whose degree is 56 and suppose that $P(n) = \frac{1}{n}$ for $n = 1, 2, 3, \ldots, 57$. Then $P(58) = \frac{a}{b}$, where *a* and *b* are relatively prime positive integers. Find $a + b$.
[Answer: 30]

Solution:
Note that $xP(x) - 1$ is a polynomial of degree 57 that has zeros $1, 2, \ldots, 57$. Therefore, $xP(x) - 1 = k(x-1)(x-2)\cdots(x-57)$ for some *k*. Comparing constant terms, we see that $-1 = k(-57!)$, so $k = 1/57!$. Substituting $x = 58$ we get

$$58P(58) - 1 = \left(\frac{1}{57!}\right)(57!) = 1$$

So $P(58) = 2/58 = 1/29$, and the answer is 30.

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Round 5: Analytic Geometry

5-1 Suppose that the point $(k+3, k-1)$ lies on the curve with equation $2y^2 - x = 6$. The larger of the two possible values of k is $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a+b$.
[Answer: 9]

Solution:
$2(k-1)^2-(k+3) = 6$
$2(k^2-2k+1)-k-3-6=0$
$2k^2-5k-7=0$
$(2k-7)(k+1) = 0$, k= 7/2 or -1 7+2 = **9**

5-2 The straight line that passes through the vertices of the parabolas $y = 2x^2 - 8x + 11$ and $y = 3x^2 - 6x + 1$ intersects the *y*-axis at the point $(0, k)$. Find $|k|$.
[Answer: 7]

Solution:
$y = 2x^2 - 8x + 11$ in vertex form $y=2(x-2)^2+3$ with vertex (2,3)
$y = 3x^2 - 6x + 1$ in vertex form $y=3(x-1)^2-2$ with vertex (1,-2) which creates the line $y=5x-7$, $|-7| = 7$

5-3 Find the number of intersection points of the graphs of $(x - \lfloor x \rfloor)^2 + y^2 = x - \lfloor x \rfloor$ and $y = \frac{1}{5}x$.
(Note: $\lfloor x \rfloor$, known as the *floor* of x, is the greatest integer less than or equal to x .)
[Answer: 11]

Solution:
For $0 \le x < 1$, $x - \lfloor x \rfloor = x$, so the given equation is $x^2 + y^2 = x$, which is a circle with center $\left(\frac{1}{2}, 0\right)$ and radius $\frac{1}{2}$. Generalizing this, if k is an integer, for $k \le x < k+1$, $x - \lfloor x \rfloor = x - k$, so the given equation is $(x-k)^2 + y^2 = x - k$, which is a circle with center $\left(k + \frac{1}{2}, 0\right)$ and radius $\frac{1}{2}$. So, the given equation represents an infinite number of circles, each of radius $\frac{1}{2}$, and with centers at ... $\left(-\frac{3}{2}, 0\right)$, $\left(-\frac{1}{2}, 0\right)$, $\left(\frac{1}{2}, 0\right)$, $\left(\frac{3}{2}, 0\right)$, … . The intersection of the line $y = \frac{1}{5}x$ with some of these circles is shown in the diagram below.

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Round 6: Trigonometry and Complex Numbers

6-1 Let $\sin 143° \cos 23° - \cos 143° \sin 23° = k$. Then $k^2 = \frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.
[Answer: 7]

Solution:

Note that the format of the expression is of the form $\sin(A - B)$ where $A = 143°$ and $B = 23°$, so $k = \sin(120°) = \frac{\sqrt{3}}{2}$, so $k^2 = \frac{3}{4}$, making the desired quantity $3 + 4 = 7$.

6-2 Suppose that $0 < \theta < \frac{\pi}{2}$ and $\cos\theta - \sin\theta = \frac{1}{2}$. Then $\cos\theta = \frac{a+\sqrt{b}}{c}$, where a, b, and c are positive integers and b is not divisible by the square of any prime number. Find $a + b + c$.
[Answer: 12]

Solution:
Note that $\cos\theta - \frac{1}{2} = \sin\theta$, and squaring both sides yields $\cos^2(A) - \cos(A) + \frac{1}{4} = \sin^2(A) = 1 - \cos^2(A)$, or $2\cos^2(A) - \cos(A) - \frac{3}{4} = 0$. Therefore $\cos(A) = \frac{1 \pm \sqrt{7}}{4}$. Since A is in Quadrant I, only the positive value is valid, so the desired quantity is $1 + 7 + 4 = 12$.

6-3 Let z be a complex number with the property that $z + \frac{400}{z}$ is a real number and $11 < z + \frac{400}{z} < 31$. Assume, further, that the real part of z is a positive integer and the imaginary part of z is positive. Find the sum of all possible values of the real part of z.
[Answer: 105]

Solution:
(In this solution, we will use the notation $\operatorname{cis}\theta$ to denote $\cos\theta + i\sin\theta$.)

Let $z = r\operatorname{cis}\theta$, with $r > 0$. Then

$$z + \frac{400}{z} = r\operatorname{cis}\theta + \frac{400}{r}\operatorname{cis}(-\theta) = \left(r + \frac{400}{r}\right)\cos\theta + \left(r - \frac{400}{r}\right)\sin\theta.$$

Since $z + \frac{400}{z}$ is real, $\operatorname{Im}\left(z + \frac{400}{z}\right) = 0$, so $\left(r - \frac{400}{r}\right)\sin\theta = 0$.
Since $\operatorname{Im} z > 0$, $\sin\theta \neq 0$, so $r - \frac{400}{r} = 0$, meaning that $r = 20$.
It follows that $z + \frac{400}{z} = \left(20 + \frac{400}{20}\right)\cos\theta = 40\cos\theta$.

We are told that $11 < z + \frac{400}{z} < 31$, so $11 < 40\cos\theta < 31$.

Note that $\text{Re}\, z = 20\cos\theta$, so it follows that $\frac{11}{2} < \text{Re}\, z < \frac{31}{2}$.
So, the possible values of $\text{Re}\, z$ are $6, 7, \ldots, 15$.

The sum of the 10 numbers in this arithmetic sequence is
$\frac{10}{2}(2 \cdot 6 + 9 \cdot 1) = 5 \cdot 21 = 105$.

Team Round

T-1 Suppose that a and b are positive integers and $a^2 - b^2 = 144$. What is the largest possible value for the greatest common divisor of a and b?
[Answer: 4]

Solution: Since $a^2 - b^2 = 144$ we have $(a-b)(a+b) = 144$. Since a and b are positive integers, so are $a-b$ and $a+b$. This means the possible values for $a-b$ and $a+b$ come from the integer factors of 144. Also note that $a-b < a+b$ since a and b are positive. So we have the following possibilities for the pairs $(a-b, a+b)$: (1,144), (2, 72), (3,48), (4, 36), (6,24), (8,18), and (9, 16). (12,12) is not an option because then we would have $b = 0$, not a positive integer. Each pair results in a system that can be solved to find a and b. For example, for (2, 72) we have:

$a - b = 2$ and $a + b = 72 \rightarrow 2a = 74 \rightarrow a = 37$ and $b = 35$. For this pair, the greatest common divisor is 1.

Proceeding in this way for each pair above we have:
(1, 144) → non-integer values for a and b
(2, 72) → $a = 37$ and $b = 35$ → gcd(a,b) = 1
(3, 48) → non-integer values for a and b
(4, 36) → $a = 20$ and $b = 16$ → gcd(a,b) = 4
(6, 24) → $a = 15$ and $b = 9$ → gcd(a,b) = 3
(8, 18) → $a = 18$ and $b = 10$ → gcd(a,b) = 2
(9, 16) → non-integer values for a and b

From these possibilities we see that the largest possible gcd is 4.

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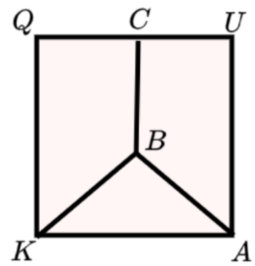
T-2 Suppose that $22x + 23y = 16$ and $23x + 22y = 29$. Find $x^2 + y^2$.
[Answer: 85]

Solution:
Add the two equations and you get 45x +45y = 45, x+y=1
Subtract the two equations and you get x-y=13
Solving for x: 2x = 14, x=7 and y = -6 with $7^2+6^2= 49+36=85$

T-3 In square $QUAK$, shown in the diagram below, $AU = 2$, $\overline{CB} \perp \overline{QU}$, and $BC = BK = AB$.



It follows that $BC = \frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.
[Answer: 9]

Solution:
Let CB=x then draw BE as an extension of CB and BE=2-x thus KE = AE = 1
$(KE)^2+(BE)^2=(BK)^2$ results in $1^2+(2-x)^2=x^2$ $\quad 1 + 4 - 4x + x^2=x^2$ $\quad$ x= 5/4, 5+4 = 9

T-4 The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 1$, and, for $n > 1$, $a_n = a_{\sqrt{n}}$ if n is a perfect square, otherwise $a_n = a_{n-1} + 1$. Find the smallest value of n for which $a_n = 103$.
[Answer: 2115]

Solution:
Let $S_k = \{a_{k^2}, a_{k^2+1}, \ldots, a_{k^2+2k}\}$, so that $S_1 = \{a_1, a_2, a_3\}$, $S_2 = \{a_4, a_5, \ldots, a_8\}$, $S_3 = \{a_9, a_{10}, \ldots, a_{15}\}$, and so on. Note that, for each k, the largest term in S_k is $a_{k^2+2k} = a_{k^2} + 2k = a_k + 2k$. So, for example, $\max(S_{30}) = a_{30^2} + 2 \cdot 30$
$= a_{30^2} + 60 = a_{30} + 60 = a_{25} + 5 + 60 = a_{25} + 65 = a_5 + 65 = a_4 + 1 + 65$
$= a_4 + 66 = a_2 + 66 = 2 + 66 = 68$.

We're looking for the smallest value of n such that $a_n = 103$. To this end, we search for the smallest value of k such that $\max(S_k)$ is 103 or more.

Investigation shows that $\max(S_k)$ is a little larger than $2k$, so we try values of k around 45, 46, or 47. We find that $\max(S_{45}) = a_{45^2+2\cdot 45} = a_{2115} = a_{45} + 90$ $= a_{36} + 99 = a_6 + 99 = a_4 + 101 = a_2 + 101 = 103.$

We have found that $a_{2115} = 103$. We will now prove that a_{2115} is the first term in the sequence with this property.

It is sufficient to prove that, for $k < 45$, $a_{k^2} + 2k < 103$, that is, $a_k + 2k < 103$. For $k < 45$, $2k < 90$, so we need to prove that, for $k < 45$, $a_k \le 103 - 90 = 13$.

Now, we find that $a_{36} = 4$. So, $\{a_{36}, a_{37}, \dots, a_{44}\} = \{4, 5, \dots, 12\}$. Also, $\max(S_5)$ $= a_5 + 10 = a_4 + 11 = a_2 + 11 = 13$, $\max(S_4) = 10$, and quick calculations show that $\max(S_3)$, $\max(S_2)$, and $\max(S_1)$ are all less than 13. So our proof is complete.

T-5 Find the area enclosed by the graph of $y = |x| + |x - 2|$ and the line $y = 8$.
[Answer: 30]

Solution: The enclosed area is a trapezoid with vertices (0,2), (2,2), (5,8) and (-3,8)
Thus the area is (1/2)(2+8)(6)= 30

T-6 Let $z = -1 + i\sqrt{3}$ and $w = -\sqrt{3} + i$. For how many integer values of n, with $2 \le n \le 100$, do w and one of the complex n^{th} roots of z have the same argument? (We assume that all arguments are written in the range $0 \le \theta < 2\pi$.)
[Answer: 8]

Solution:
Note that $-1 + i\sqrt{3}$ has an argument of $\frac{2\pi}{3}$, and $-\sqrt{3} + i$ has an argument of $\frac{5\pi}{6}$. Setting the argument of an nth root of the given complex number equal to $\frac{5\pi}{6}$ yields $\frac{2\pi}{3n} + \frac{2k\pi}{n} = \frac{5\pi}{6}$, or $4 + 12k = 5n$, or $5n - 12k = 4$. Note that $(8,3)$ is a solution for (n, k), and to find the total number of solutions, we have $8 + 12(N - 1) \le 100$. This yields $N \le \frac{26}{3}$, making 8 the desired answer.