# 2024 NEAML

## SOLUTIONS

**Round 1 - Arithmetic and Number Theory**

**Problem 1. Compute the ordered pair of positive integers $(x, y)$ with $10 \le x \le 20$ and $10 \le y \le 20$ for which $63x - 44y = 8$.**

**Solution $\boxed{(\mathbf{12}, \mathbf{17})}$ The problem statement implies $63x - 8 = 44y \to y = \frac{63x-8}{44}$, which implies $y = x + \frac{19x-8}{44}$. Because $x \in [10, 20]$, try different values of $x$ to see if $y$ is an integer. Notice that $x = 10$ and $x = 11$ do not generate integer values of $y$, but $x = 12$ implies that $y = 12 + \frac{220}{44} = 17$, and so the answer is $(\mathbf{12}, \mathbf{17})$.**

**Problem 2. Compute the number of positive integers less than 2022 that can be expressed as the difference of the squares of two consecutive integers.**

**Solution $\boxed{1011}$ The problem statement implies that for some integer $n$, $0 < (n+1)^2 - n^2 < 2022$. This implies $0 < 2n+1 < 2022$, so $-0.5 < n < 1010.5$. Thus the integers $n$ that work are $\{0, 1, 2, \ldots, 1010\}$. There are therefore 1011 positive integers less than 2022 that can be expressed as the difference of the squares of two consecutive integers.**

**Problem 3. Three positive integers are in increasing arithmetic progression. If the middle integer is decreased by 40, then the new three integers are in geometric progression. If instead the third integer is increased by 320, then the new three integers are in geometric progression. Compute the first integer.**

**Solution $\boxed{20}$ Call the three positive integers $a - d$, $a$, and $a + d$.**

**The problem statement implies that $a-d$, $a-40$, and $a+d$ are in geometric progression, which implies $\frac{a-40}{a-d} = \frac{a+d}{a-40} \to d^2 - 80a + 1600 = 0$.**

**The problem statement also implies that $a - d$, $a$, and $a + d + 320$ are in geometric progression. This implies $\frac{a}{a-d} = \frac{a+d+320}{a} \to d^2 = 320a - 320d$.**

**The equations from the last two paragraphs imply $80a - 1600 = 320a - 320d \to a = \frac{4d-20}{3}$. Substituting into $d^2 - 80a + 1600 = 0$ obtains $3d^2 - 320d + 6400 = 0$. Factor to obtain $(3d - 80)(d - 80) = 0$, which has only one integer solution, namely $d = 80$. Therefore $a = 100$, and so $a - d = 100 - 80 = \mathbf{20}$.**

**Round 2 - Algebra 1**

**Problem 1. Dora works at twice Diego's pace. If Diego worked three more hours than it would take Dora to do a job, then Diego would finish 75% of the job. Compute the number of hours it would take Dora to do the job.**

**Solution $\boxed{6}$ Let $D$ be the number of hours Diego takes to do a job. Then it follows that $D\left(\frac{1}{2D} + 3\right) = \frac{3}{4}$, which solves to obtain $\frac{1}{2} + 3D = \frac{3}{4} \to D = \frac{1}{12}$. Thus it takes Diego 12 hours to do the job, so it takes Dora 6 hours to do the job.**

**Problem 2. Compute all $x$ such that $(2x - 3)^2 + (3x - 5)^2 = (5x - 8)^2$.**

**Solution $\boxed{\frac{3}{2} \text{ and } \frac{5}{3} \textit{ (need both)}}$ The given equation is of the form $A^2 + B^2 = (A + B)^2$, which implies $A^2 + B^2 = A^2 + B^2 + 2AB \to AB = 0$. Thus the solutions of the given equation are the solutions of $2x - 3 = 0$ and $3x - 5 = 0$. Thus the desired values of $x$ are $\frac{3}{2}$ and $\frac{5}{3}$.**

**Problem 3.** **Compute** $\dfrac{2023^3 + 9 \cdot 2023^2 + 23 \cdot 2023 + 15}{2023^3 - 1992 \cdot 2023^2 - 15985 \cdot 2023 - 30000}$.

**Solution** $\boxed{\mathbf{88}}$ **Let** $x = 2023$. **Then the numerator factors as** $(x+1)(x+3)(x+5)$, **which has zeroes** $x = -1$, $x = -3$, **and** $x = -5$. **Notice that** $-3$ **and** $-5$ **are also zeroes of the denominator. Factor or use synthetic division to obtain that the denominator factors as** $(x+3)(x+5)(x-2000)$. **Thus the given expression is equivalent to** $\dfrac{2023+1}{2023-2000} = \dfrac{2024}{23} = \mathbf{88}$.

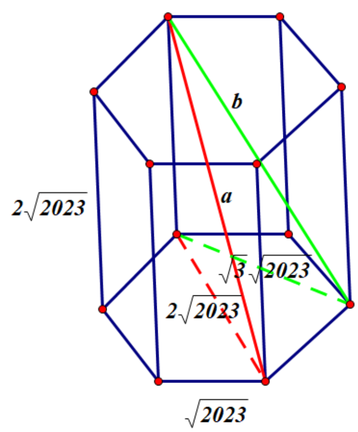
**Round 3 - Geometry**

**Problem 1. In a field, a square grassy area 100 feet by 100 feet is surrounded by a fence. Two horses walk on different paths (one inside the fenced area and one outside the fenced area), each staying 10 feet from the fence at all times. Given that the greatest possible distance between the horses is** $M$ **feet, compute the integer closest to** $M$.

**Solution** $\boxed{\mathbf{137}}$ **The horse inside the fenced area walks along a square of side length** $80$ **feet. The horse outside the fenced area walks along a "rectangle" with quarter-circles at the corners. The greatest possible distance between the horses will occur when the horse inside the fenced area is at a corner of its path and the horse outside the fenced area is at the midpoint of the quarter-circle opposite the first horse. In this case, the greatest possible distance is the length of the hypotenuse of an isosceles right triangle whose leg measures** $90 + (10/\sqrt{2}) = 90 + 5\sqrt{2}$ **feet, so** $M = (90 + 5\sqrt{2})(\sqrt{2}) = 90\sqrt{2} + 10$, **which is approximately 137 feet.**

**Problem 2. A right prism has regular hexagons with side length** $\sqrt{2023}$ **for its bases and six congruent rectangles of area 4046 each for its lateral faces. There are two noncongruent space diagonals in the prism, and these space diagonals have lengths** $a$ **and** $b$. **Compute** $|a^2 - b^2|$.

**Solution** $\boxed{\mathbf{2023}}$ **Note that the height of the prism is** $4046 \div \sqrt{2023} = 2\sqrt{2023}$. **Consider the diagram below, which shows the hexagonal prism and the two space diagonals.**




**By properties of hexagons, the dashed segments have the lengths shown. (The reader can verify these measurements with basic trigonometry.) By the Pythagorean Theorem, it follows that** $(2\sqrt{2023})^2 + (2\sqrt{2023})^2 = a^2 \to a^2 = 8(2023)$ **and** $(2\sqrt{2023})^2 + (\sqrt{3} \cdot \sqrt{2023})^2 = b^2 \to b^2 = 7(2023)$. **Thus it follows that** $a^2 - b^2 = \mathbf{2023}$.

**Problem 3.** In $\triangle TRI$, $TR = 20$, $TI = 18$, and $RI = 12$. Point $A$ is on $\overline{TI}$ such that $\overline{RA}$ bisects $\angle TRI$. Point $N$ is on $\overline{RA}$ such that $\overline{IN}$ bisects $\angle TIR$. Given that $\frac{RN}{NA}$ is $\frac{p}{q}$ where $p$ and $q$ are whole numbers and the fraction is in lowest terms, compute $p+q$.

**Solution** $\boxed{25}$ By the angle bisector theorem applied to $\triangle TRI$, it follows that $\frac{TA}{AI} = \frac{TR}{RI} = \frac{20}{12}$, which implies $AI = \frac{27}{4}$. By the angle bisector theorem applied to $\triangle AIR$, it follows that $\frac{RN}{NA} = \frac{RI}{IA} = \frac{12}{27/4}$, which implies $\frac{RN}{NA} = \frac{48}{27} = \frac{16}{9}$, so the answer is $16+9 = \mathbf{25}$.

**Round 4 - Algebra 2**

**Problem 1.** Compute $x$ such that $\sqrt{x-3} + x = 15$.

**Solution** $\boxed{12}$ The given equation is equivalent to $\sqrt{x-3} = 15 - x$, which implies $x - 3 = 225 - 30x + x^2 \rightarrow x^2 - 31x + 228 = 0$. Factor to obtain $(x-19)(x-12) = 0$. There are two solutions: $x = 19$ (which does not check) and $x = \mathbf{12}$ (which does check).

**Problem 2.** Suppose that $m$ and $n$ are positive numbers such that $\frac{(m+n)^3}{m^3+n^3} = \frac{14}{5}$. Compute $\frac{m^2+n^2}{mn}$.

**Solution** $\boxed{\frac{8}{3}}$ Note that $\frac{(m+n)^3}{m^3+n^3} = \frac{m^3+n^3+3mn(m+n)}{m^3+n^3} = 1 + \frac{3mn(m+n)}{(m+n)(m^2-mn+n^2} = \frac{14}{5}$, which implies $\frac{3mn}{m^2-mn+n^2} = \frac{9}{5}$. Take reciprocals on both sides to obtain $\frac{m^2-mn+n^2}{3mn} = \frac{m^2+n^2}{3mn} - \frac{1}{3} = \frac{5}{9}$, which implies $\frac{m^2+n^2}{3mn} = \frac{8}{9}$, so it follows that $\frac{m^2+n^2}{mn} = 3 \cdot \frac{8}{9} = \mathbf{\frac{8}{3}}$.

**Problem 3.** Given that $x^2 + 3x + 9 = 0$, compute the value of $x^6$.

**Solution** $\boxed{729}$ Notice that $(x-3)(x^2+3x+9) = x^3 - 27 = 0$, so $x^6 = (x^3)^2 = 27^2 = \mathbf{729}$.

**Round 5 - Analytic Geometry**

**Problem 1.** In the coordinate plane, the image of the point $(20, 21)$ after a reflection in the line $y = x$ is $(A, B)$. The image of $(A, B)$ after a reflection in the line $y = 2x$ is $(C, D)$. The image of $(C, D)$ after a reflection in the line $y = -\frac{1}{2}x$ is $(E, F)$. Compute $E + F$.

**Solution** $\boxed{-41}$ The point $(A, B)$ is $(21, 20)$ because reflecting over $y = x$ reverses the coordinates. Then, notice that the lines $y = 2x$ and $y = -\frac{1}{2}x$ are perpendicular and cross at $(0, 0)$. Therefore, a composition of reflections over those two lines is a half-turn centered at $(0, 0)$. Thus the ordered pair $(E, F)$ is $(-21, -20)$. The sum $E+F$ is $\mathbf{-41}$.

**Problem 2.** Octagon $MOUSEPAD$ has vertices $M(0, 4)$, $O(0, 0)$, $U(7, 0)$, $S(7, 1)$, $E(2, 1)$, $P(2, 3)$, $A(1, 3)$, and $D(1, 4)$. The line $y = kx$ divides $MOUSEPAD$ into two regions of equal area. Compute $k$.

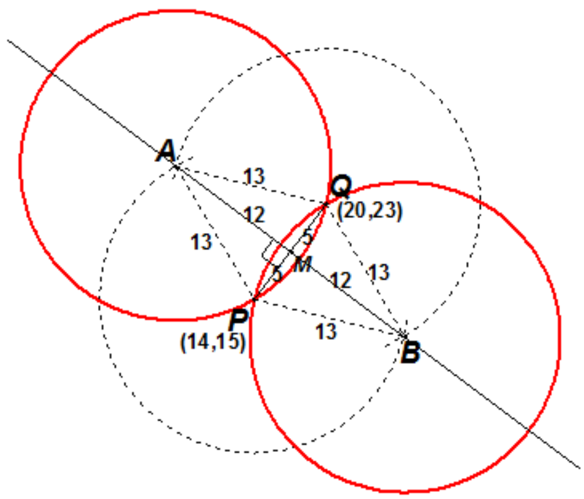
**Solution** $\boxed{\frac{1}{2}}$ The area of $MOUSEPAD$ is 12, which can be obtained by counting $1 \times 1$ blocks. The five $1 \times 1$ blocks on the right side of $MOUSEPAD$ will combine with a triangle of area 1 and base 2 to create a 6-square-unit shape. Solving $\frac{1}{2} \cdot 2 \cdot h = 1$ yields $h = 1$, so the line $y = kx$ passes through $(2, 1)$. Thus the slope is $\mathbf{\frac{1}{2}}$.

**Problem 3.** The points $P(14, 15)$ and $Q(20, 23)$ each lie on two distinct circles with radius

**13. Compute the distance between the centers of the two circles.**

**Solution** $\boxed{24}$ **Let the center of one circle be** $(h, k)$**. Then it follows that** $(h-14)^2 + (k-15)^2 = 169$ **and** $(h-20)^2 + (k-23)^2 = 169$**. Expand and subtract to obtain** $12h - 204 + 16k - 304 = 0 \to 3h + 4k = 127$**, so** $k = \frac{127-3h}{4}$**. Substitute to obtain** $(h-14)^2 + \left(\frac{127-3h}{4} - 15\right)^2 = 169$**. This solves to obtain** $h = \frac{37}{5}$ **or** $h = \frac{133}{5}$**, which implies** $k = \frac{131}{5}$ **or** $k = \frac{59}{5}$**. The distance between these possible centers is** $\sqrt{\left(\frac{96}{5}\right)^2 + \left(\frac{72}{5}\right)^2} = \sqrt{\left(\frac{120}{5}\right)^2} = \mathbf{24}$**.**

***Alternate Solution:*** **Consider the diagram.**




**The centers of the circles (labeled** $A$ **and** $B$ **in the diagram) must lie on the perpendicular bisector of** $\overline{PQ}$ **because they are equidistant from its endpoints. Label the midpoint of** $\overline{PQ}$ **as** $M$ **and draw the right** $\triangle AMP$ **with hypotenuse** $AP = 13$**. Use the distance formula or the Pythagorean Theorem to find that** $PQ = 10$**, which implies** $PM = 5$**. This implies that there is a** $5-12-13$ **Pythagorean triple, so** $AM = 12$ **and** $AB = 2(12) = 24$**.**

**Round 6 - Trig and Complex Numbers**

**Problem 1. Compute the least positive integer** $N$ **for which** $\cos(12N)^\circ$ **and** $\sin(5N)^\circ$ **are both negative.**

**Solution** $\boxed{38}$ **The least positive values of** $N$ **for which** $\cos(12N)^\circ$ **is negative satisfy** $90 < 12N < 270 \to 7.5 < N < 22.5$**. The least positive values of** $N$ **for which** $\sin(5N)^\circ$ **is negative satisfy** $180 < 5N < 360 \to 36 < N < 72$**. There is no overlap in these intervals, so consider the next least values of** $N$ **that satisfy** $\cos(12N)^\circ < 0$**, which are** $450 < 12N < 630 \to 37.5 < N < 52.5$**. The least** $N$ **in both of these latter intervals is 38.**

**Problem 2. Let** $i = \sqrt{-1}$**. Suppose that** $z = 5 + bi$ **for some complex number** $z$ **and some positive integer** $b$**. Given that the imaginary part of** $z^3$ **is 8 more than the imaginary part of** $z^2$**, compute** $b$**.**

**Solution** $\boxed{8}$ **Note that** $z^2 = (25 - b^2) + 10bi$ **and** $z^3 = (125 - 15b^2) + (75b - b^3)i$**. The problem statement implies that** $75b - b^3 = 8 + 10b \to b^3 - 65b + 8 = 0$**. By the Rational Root Theorem, the set of possible integer roots** $b$ **is** $\{1, 2, 4, 8\}$**. By inspection,** $b = \mathbf{8}$**.**

**Problem 3.** **Suppose** $A$ **and** $B$ **are first-quadrant angles with** $\cos A = \frac{3}{5}$ **and** $\cos(A+B) = \frac{13}{25}$**. Given that** $\cos B$ **can be expressed in simplest form as** $\dfrac{K + 8\sqrt{M}}{N}$ **where** $K$, $M$, **and** $N$ **are whole numbers, compute** $K + M + N$.

**Solution** $\boxed{278}$ **Using the formula for the cosine of an angle sum and also a Pythagorean identity, it follows that** $\frac{3}{5}\cos B - \frac{4}{5}\sqrt{1-\cos^2 B} = \frac{13}{25}$**. This implies** $15\cos B - 13 = 20\sqrt{1-\cos^2 B} \rightarrow 225\cos^2 B - 390\cos B + 169 = 400 - 400\cos^2 B$**. Solve** $625\cos^2 B - 390\cos B - 231 = 0$ **for the positive value of** $\cos B$ **to obtain** $\cos B = \dfrac{39 + 8\sqrt{114}}{125}$**, so the answer is** $39 + 114 + 125 =$ **278.**

**NEAML TEAM ROUND**

**Problem 1.** **Suppose that** $a$, $b$, **and** $c$ **are real numbers such that** $a$ **is** $60\%$ **larger than** $c$ **and** $b$ **is** $20\%$ **larger than** $c$**. Given that** $a$ **is** $p\%$ **larger than** $b$**, compute the integer closest to** $p$.

**Solution 33 The problem statement implies that** $a = 1.6c$ **and** $b = 1.2c$**. Thus,** $\frac{a}{b} = \frac{1.6}{1.2} = \frac{16}{12} = \frac{4}{3} \approx 1.33$**, so** $100 + p \approx 133 \rightarrow p \approx$ **33.**

**Problem 2.** **The numbers** $g_1$, $g_2$, **and** $g_3$ **with** $g_1 < g_2 < g_3$ **are in geometric sequence. Given that** $g_1 + g_2 = 60$ **and** $g_3 - g_2 = 56$**, compute** $g_2$.

**Solution** $\boxed{42}$ **Let** $g_2 = rg_1$ **and** $g_3 = r^2 g_1$**. Then the given equations are equivalent to** $rg_1 + g_1 = 60 \rightarrow g_1(r+1) = 60$ **and** $r^2 g_1 - rg_1 = 56 \rightarrow rg_1(r-1) = 56$**. By division, this implies** $\dfrac{r(r-1)}{r+1} = \dfrac{56}{60} = \dfrac{14}{15}$**. Cross-multiply to obtain** $15r^2 - 15r = 14r + 14 \rightarrow 15r^2 - 29r - 14 = 0$**. Factoring yields** $(3r-7)(5r+2) = 0$**, and this gives** $r = 7/3$ **or** $r = -2/5$**. Because** $g_1 < g_2$**, reject** $r = -2/5$**, so** $r = 7/3$ **which implies** $g_1 = 18$ **and** $g_2 = 18 \cdot \frac{7}{3} =$ **42.**

**Problem 3.** **Compute the least positive integer** $n$ **such that** $24 \cdot n$ **has exactly** $24$ **positive integer divisors.**

**Solution** $\boxed{15}$ **Notice that** $24 = 2^3 \cdot 3$**. Therefore the least** $n$ **will be the number such that** $24 \cdot n = 2^{3+\alpha} \cdot 3^{1+\beta} \cdot 5^{\gamma}$**. Thus it follows that** $(4+\alpha)(2+\beta)(1+\gamma) = 24$.

**If** $\alpha = 0$**, then** $(2+\beta)(1+\gamma) = 6 \rightarrow \beta = 1$ **and** $\gamma = 1$ **or** $\beta = 4$ **and** $\gamma = 0$ **or** $\beta = 0$ **and** $\gamma = 5$**. These correspond to** $n = 15$ **or** $n = 81$ **or** $n = 3125$**. The least of these is** $n = 15$**. Note that** $\alpha \neq 1$ **because 24 is not a multiple of 5. If** $\alpha = 2$**, then** $(2+\beta)(1+\gamma) = 4 \rightarrow \beta = 2$ **and** $\gamma = 0$ **or** $\beta = 0$ **and** $\gamma = 1$**. These correspond to** $n = 36$ **or** $n = 20$**. Neither of these** $n$ **is less than** $15$.

**Problem 4.** **Compute the number of integers** $N$**, with** $1 \leq N \leq 2023$**, that contain at least one digit that is a 0 or a 2 or a 3.**

**Solution** $\boxed{1281}$ **Proceed by complementary counting. The numbers that have none of the digits in** $\{0, 2, 3\}$ **have digits chosen from the set of the seven other digits. This implies that there are** $7$ **one-digit numbers,** $7^2 = 49$ **two-digit numbers, and** $7^3 = 343$ **three-digit numbers that don't contain a 0 or 2 or 3. Further, there are an additional** $7^3 = 343$ **numbers between 1000 and 1999 that don't contain a 0 or 2 or 3, and every number between 2000 and 2023 has a 2, so there are** $2023 - 343 - 343 - 49 - 7 =$ **1281 numbers in the given interval that contain at least one digit that is a 0 or 2 or 3.**

**Problem 5.** **Sophia is graphing** $f(x) = ax^2 + bx + c$**. She graphs five points whose** $x$**-coordinates are five consecutive integers. Her** $y$**-coordinates are** $15$, $23$, $37$, $51$, **and** $71$**. Emma points out that one of Sophia's** $y$**-coordinates is incorrect and that it should be**

$m$. **Compute** $m$.

**Solution The differences between the** $y$**-coordinates are** $8, 14, 14,$ **and** $20$**. The "second differences" are** $6, 0,$ **and** $6$**. This is not possible for an increasing quadratic function; the second differences should be equal. They would be equal if the first differences in the middle of the sequence were** $12$ **and** $16$ **instead of** $14$ **and** $14$**. Thus the first differences should be** $8, 12, 16,$ **and** $20$**, which generate common second differences of 4. Thus the correct third** $y$**-coordinate is** $23 + 12 = 51 - 16 =$ **35.**

**Problem 6. Compute the area bounded below by the graph of** $y = |x - 1| + |2x - 4|$ **and above by the graph of** $y = 5$**.**

**Solution** $\boxed{\mathbf{\frac{23}{3}} \text{ or } \mathbf{7\frac{2}{3}}}$ **The graph of** $y = |x - 1| + |2x - 4|$ **is composed of parts of three lines, one for** $x < 1$**, one for** $1 \leq x \leq 2$**, and one for** $x > 2$**. (One can determine this by setting each expression in absolute value bars equal to zero.) Thus, the key points on the graph of** $y = |x - 1| + |2x - 4|$ **to consider are** $(0, 5)$**,** $(1, 2)$**,** $(2, 1)$**, and** $(10/3, 5)$**. By dropping perpendiculars from** $(1, 2)$ **and** $(2, 1)$ **to the graph of** $y = 5$**, dissect the region into two triangles and a trapezoid. The area of the region is the sum of these three areas:** $\frac{1}{2}(1)(3) + \frac{1}{2}(1)(3 + 4) + \frac{1}{2}(4/3)(4) = \frac{3}{2} + \frac{7}{2} + \frac{8}{3} = \mathbf{\frac{23}{3}}$**.**