CT ARML Team, 2024

Team Selection Test 1

1. Triangle ABC has side lengths $AB = 12$, $BC = 15$, and $CA = 13$. Points X and Y are placed on sides $\overline{AB}$ and $\overline{AC}$, respectively, in such a way that $\angle AXY = \angle ACB$. If $XY = 6$, what is $AX + AY$?
   [Answer: 10]

2. Compute the area of the region consisting of all points (x, y) that satisfy the inequalities

$$\min\{|x|, |y|\} \leq 20 \qquad \text{and} \qquad |x| + |y| \leq 120$$

   [Answer: 16000]

3. Suppose that a and b are nonzero constants, and that the zeros of $x^2 + ax + b$ are exactly double the zeros of $bx^2 - ax + a$. Then $a = (-1)^m p$ and $b = (-1)^n q$, where p and q are positive integers, m is 0 or 1, and n is 0 or 1. Find $p + q + m + n$.
   [Answer: 4]

4. Compute the smallest positive integer n such that $n^2 + 7n + 89$ is a multiple of 77.
   [Answer: 18]

5. Let a be the integer such that $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{22} + \frac{1}{23} = \frac{a}{23!}$. Compute the remainder when a is divided by 13.
   [Answer: 7]

6. How many ways are there to write 2024 as the sum of three positive even integers, where different orders of addition are counted distinctly?
   [Answer: 510555]

7. Let ABC be an isosceles triangle with $AB = AC$ and $\frac{\cos B}{\cos A} = \frac{15}{7}$. Then $\frac{\sin B}{\sin A} = \frac{p}{q}$, where p and q are relatively prime positive integers. Find $2p + q$.
   [Answer: 16]

8. How many ordered pairs of complex numbers (z, w) satisfy the equations $z^{19}w^{25} = 1$, $z^5w^7 = 1$, and $z^4 + w^4 = 2$?
[Answer: 4]

9. Compute $\frac{1}{(\sin 10°)(\sin 50°)(\sin 70°)}$.
[Answer: 8]

10. Austin will make a five-flight random journey among the airports at Indianapolis, Jackson, Kansas City, Lincoln, and Milwaukee. He will start at Lincoln, and for each flight he will select randomly and uniformly among the four airports different from the one at which he is currently located. Given that the destination of his fifth flight is Lincoln, the probability that he visits all five airports during his journey is $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.
[Answer: 19]

Solutions to TST 1

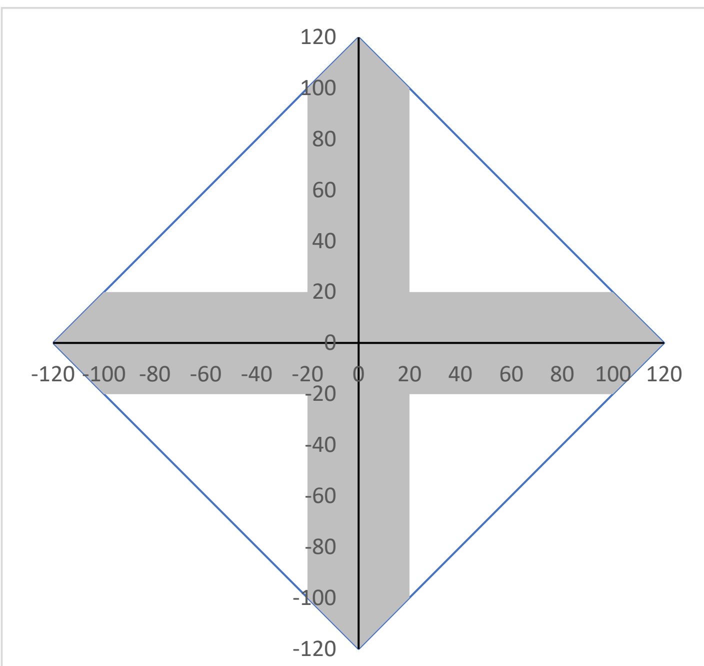
Solution for Problem 1

$\angle AXY = \angle ACB$ implies $\triangle AXY \sim \triangle ACB$. Then:

$$\frac{AY}{12} = \frac{AX}{13} = \frac{6}{15}.$$

$$AX + AY = \frac{6 \cdot 12}{15} + \frac{6 \cdot 13}{15} = 10.$$

Solution for Problem 2



The region (shaded gray) is the big square minus four triangles. The area is therefore:

$$120 \cdot 120 \cdot 2 - 80 \cdot 80 \cdot 2 = 16000.$$

Solution for Problem 3

Let u, v be the zeros of $bx^2 - ax + a$, then $2u, 2v$ are the zeros of $x^2 + ax + b$. We have:

$$\begin{cases} (2u)(2v) = b \\ 2u + 2v = -a \end{cases}, \begin{cases} uv = a/b \\ u + v = a/b \end{cases}.$$

Eliminating u and v, we have:

$$\begin{cases} b = 4a/b \\ -a = 2a/b \end{cases}$$

The solutions are $a = 1, b = -2$. Therefore $p + q + m + n = 4$.

Solution for Problem 4

In order for $n^2 + 7n + 89$ to be a multiple of 7, we need $n^2 \equiv 2 \bmod 7$, which requires n to be 3 or 4 $\bmod 7$.

In order for $n^2 + 7n + 89$ to be a multiple of 11, n has to be 2, 7 or 8 $\bmod 11$.

The smallest n that satisfies both conditions is therefore 18.

Solution for Problem 5

Using the common denominator $23!$, the n-th term in the sum has numerator $\frac{23!}{n}$, which is divisible by 13 except when $n = 13$. Thus we can set

$$a = (1 \cdot 2 \cdots 12) \cdot (14 \cdot 15 \cdots 23) \equiv (1 \cdot 2 \cdots 12) \cdot (1 \cdot 2 \cdots 10) \bmod 13.$$

We can take the cumulative product in mod-13 arithmetic, getting $(1 \cdot 2 \cdots 10) \equiv 6 \bmod 13$ and $(1 \cdot 2 \cdots 12) \equiv 12 \bmod 13$. Therefore the answer is $6 \cdot 12 \equiv 7 \bmod 13$.

Solution for Problem 6

We reduce the problem to equivalent ones through several steps:

1. Dividing all numbers by 2, the problem is equivalent to finding the number of ways to write 1012 as the sum of 3 positive integers.

2. Subtracting 1 from all 3 summands, the problem is equivalent to finding the number of ways to write 1009 as the sum of 3 non-negative integers.

3. Imagine that there are 1009 white balls in a line, with 2 black balls placed between them to separate them into 3 groups. The problem is equivalent to finding the number of ways to select 2 balls out of 1011 to paint black.

The answer is therefore $C(1011,2) = 510{,}555$.

Solution for Problem 7

Let D be the middle point of $\overline{BC}$ and $\theta = \angle BAD$. Then B and θ are complementary angles. We have:

$$\cos A = \cos(2\theta) = 1 - 2(\cos B)^2.$$

Therefore from $\frac{\cos B}{\cos A} = \frac{15}{7}$, we have $\frac{\cos B}{1-2(\cos B)^2} = \frac{15}{7}$, or $30(\cos B)^2 + 7(\cos B) - 15 = 0$. The solution of this equation is $\cos B = \frac{3}{5}$.

Now $\sin A = \sin(2\theta) = 2(\sin\theta)(\cos\theta) = 2(\cos B)(\sin B)$, therefore

$$\frac{\sin B}{\sin A} = \frac{\sin B}{2(\cos B)(\sin B)} = \frac{1}{2} \cdot \frac{5}{3} = \frac{5}{6}.$$

The answer is $2 \cdot 5 + 6 = 16$.

Solution for Problem 8

From $z^{14}w^{18} = 1$ and $z^5w^7 = 1$, we get $z^4w^4 = 1$. Solving this together with $z^4 + w^4 = 2$, we get $z^4 = 1$ and $w^4 = 1$. Therefore $z, w \in \{1, -1, i, -i\}$. With each possible choice of z, we see that $z^5w^7 = 1$ implies $z = w$. Therefore the answer is 4.

Solution for Problem 9

First:

$$(\sin 10^\circ)(\sin 50^\circ) = \sin(30^\circ - 20^\circ)\sin(30^\circ + 20^\circ)$$

$$= \big((\sin 30^\circ)(\cos 20^\circ) + (\cos 30^\circ)(\sin 20^\circ)\big)\big((\sin 30^\circ)(\cos 20^\circ) - (\cos 30^\circ)(\sin 20^\circ)\big)$$

$$= (\sin 30^\circ)^2(\cos 20^\circ)^2 - (\cos 30^\circ)^2(\sin 20^\circ)^2$$

$$= \frac{1}{4}(\cos 20^\circ)^2 - \frac{3}{4}[1 - (\cos 20^\circ)^2]$$

$$= \frac{4(\cos 20^\circ)^2 - 3}{4}.$$

Then:

$$(\sin 10^\circ)(\sin 50^\circ)(\sin 70^\circ)$$

$$= \frac{4(\cos 20°)^2 - 3}{4} \cdot (\cos 20°)$$

$$= \frac{4(\cos 20°)^3 - 3(\cos 20°)}{4}.$$

Since $\cos(3x) = 4(\cos x)^3 - 3(\cos x)$, we have

$$(\sin 10°)(\sin 50°)(\sin 70°) = \frac{\cos 60°}{4} = \frac{1}{8}.$$

Taking the inverse, the answer is 8.

<u>Solution for Problem 10</u>

The total number of valid five-flight journeys that start and end at Lincoln is equal to $4^4 - \left(4^3 - (4^2 - 4)\right) = 204.$ We get this by counting four legs each with four independent choices, minus those that end at Lincoln at the end of four legs, and applying recursion.

The number of journeys that visit all five airports is equal to $4! = 24$.

Therefore, the probability is $\frac{24}{204} = \frac{2}{17}$, with $2 + 17 = 19$.

Relay 1-1. Let $a = 19$, $b = 20$, and $c = 21$. Compute

$$\frac{a^2 + b^2 + c^2 + 2ab + 2bc + 2ca}{a + b + c}.$$

Relay 1-2. Let $T = \text{TNYWR}$. Lydia is a professional swimmer and can swim one-fifth of a lap of a pool in an impressive 20.19 seconds, and she swims at a constant rate. Rounded to the nearest integer, compute the number of minutes required for Lydia to swim T laps.

Relay 1-3. Let $T = \text{TNYWR}$. In $\triangle ABC$, $m\angle C = 90^\circ$ and $AC = BC = \sqrt{T - 3}$. Circles O and P each have radius r and lie inside $\triangle ABC$. Circle O is tangent to $\overline{AC}$ and $\overline{BC}$. Circle P is externally tangent to circle O and to $\overline{AB}$. Given that points C, O, and P are collinear, compute r.

Answer 1-1. 60

Answer 1-2. 101

Answer 1-3. $3 - \sqrt{2}$

Relay 1-1. Let $a = 19$, $b = 20$, and $c = 21$. Compute

$$\frac{a^2+b^2+c^2+2ab+2bc+2ca}{a+b+c}.$$

Solution 1-1. Note that the numerator of the given expression factors as $(a+b+c)^2$, hence the expression to be computed equals $a+b+c = 19+20+21 = \mathbf{60}$.

Relay 1-2. Let T = TNYWR. Lydia is a professional swimmer and can swim one-fifth of a lap of a pool in an impressive 20.19 seconds, and she swims at a constant rate. Rounded to the nearest integer, compute the number of <u>minutes</u> required for Lydia to swim T laps.

Solution 1-2. Lydia swims a lap in $5 \cdot 20.19 = 100.95$ seconds. The number of minutes required for Lydia to swim T laps is therefore $100.95 \cdot T/60$. With $T = 60$, the desired number of minutes, rounded to the nearest integer, is $\mathbf{101}$.

Relay 1-3. Let T = TNYWR. In $\triangle ABC$, $m\angle C = 90°$ and $AC = BC = \sqrt{T-3}$. Circles O and P each have radius r and lie inside $\triangle ABC$. Circle O is tangent to $\overline{AC}$ and $\overline{BC}$. Circle P is externally tangent to circle O and to $\overline{AB}$. Given that points C, O, and P are collinear, compute r.

Solution 1-3. Let A' and B' be the respective feet of the perpendiculars from O to $\overline{AC}$ and $\overline{BC}$. Let H be the foot of the altitude from C to $\overline{AB}$. Because $\triangle ABC$ is isosceles, it follows that $A'OB'C$ is a square, $m\angle B'CO = 45°$, and $m\angle BCH = 45°$. Hence H lies on the same line as C, O, and P. In terms of r, the length CH is $CO + OP + PH = r\sqrt{2} + 2r + r = (3+\sqrt{2})r$. Because $AC = BC = \sqrt{T-3}$, it follows that $CH = \frac{\sqrt{T-3}}{\sqrt{2}}$. Thus $r = \frac{\sqrt{T-3}}{\sqrt{2}(3+\sqrt{2})} = \frac{(3\sqrt{2}-2)\sqrt{T-3}}{14}$. With $T = 101$, $\sqrt{T-3} = \sqrt{98} = 7\sqrt{2}$, and it follows that $r = \mathbf{3-\sqrt{2}}$.

1 Team Problems

Problem 1. The points $(1, 2, 3)$ and $(3, 3, 2)$ are vertices of a cube. Compute the product of all possible distinct volumes of the cube.

Problem 2. Eight students attend a Harper Valley ARML practice. At the end of the practice, they decide to take selfies to celebrate the event. Each selfie will have either two or three students in the picture. Compute the minimum number of selfies so that each pair of the eight students appears in exactly one selfie.

Problem 3. Compute the least positive value of t such that

$$\operatorname{Arcsin}(\sin(t)),\ \operatorname{Arccos}(\cos(t)),\ \operatorname{Arctan}(\tan(t))$$

form (in some order) a three-term arithmetic progression with a nonzero common difference.

Problem 4. In non-right triangle ABC, distinct points P, Q, R, and S lie on $\overline{BC}$ in that order such that $\angle BAP \cong \angle PAQ \cong \angle QAR \cong \angle RAS \cong \angle SAC$. Given that the angles of $\triangle ABC$ are congruent to the angles of $\triangle APQ$ in some order of correspondence, compute $\mathrm{m}\angle B$ in degrees.

Problem 5. Consider the system of equations

$$\begin{aligned}\log_4 x + \log_8(yz) &= 2\\ \log_4 y + \log_8(xz) &= 4\\ \log_4 z + \log_8(xy) &= 5.\end{aligned}$$

Given that xyz can be expressed in the form 2^k, compute k.

Problem 6. A complex number z is selected uniformly at random such that $|z| = 1$. Compute the probability that z and z^{2019} both lie in Quadrant II in the complex plane.

Problem 7. Compute the least positive integer n such that the sum of the digits of n is five times the sum of the digits of $(n + 2019)$.

Problem 8. Compute the greatest real number K for which the graphs of

$$(|x| - 5)^2 + (|y| - 5)^2 = K \quad \text{and} \quad (x - 1)^2 + (y + 1)^2 = 37$$

have exactly two intersection points.

Problem 9. To *morph* a sequence means to replace two terms a and b with $a+1$ and $b-1$ if and only if $a+1 < b-1$, and such an operation is referred to as a morph. Compute the least number of morphs needed to transform the sequence $1^2, 2^2, 3^2, \ldots, 10^2$ into an arithmetic progression.

Problem 10. Triangle ABC is inscribed in circle ω. The tangents to ω at B and C meet at point T. The tangent to ω at A intersects the perpendicular bisector of $\overline{AT}$ at point P. Given that $AB = 14$, $AC = 30$, and $BC = 40$, compute $[PBC]$.

2 Answers to Team Problems

Answer 1. 216

Answer 2. 12

Answer 3. $\dfrac{3\pi}{4}$

Answer 4. $\dfrac{45}{2}$ (or $22\frac{1}{2}$ or 22.5)

Answer 5. $\dfrac{66}{7}$ (or $9\frac{3}{7}$)

Answer 6. $\dfrac{505}{8076}$

Answer 7. 7986

Answer 8. 29

Answer 9. 56

Answer 10. $\dfrac{800}{3}$ (or $266\frac{2}{3}$ or $266.\overline{6}$)

3 Solutions to Team Problems

Problem 1. The points $(1, 2, 3)$ and $(3, 3, 2)$ are vertices of a cube. Compute the product of all possible distinct volumes of the cube.

Solution 1. The distance between points $A(1, 2, 3)$ and $B(3, 3, 2)$ is $AB = \sqrt{(3-1)^2 + (3-2)^2 + (2-3)^2} = \sqrt{6}$. Denote by s the side length of the cube. Consider three possibilities.

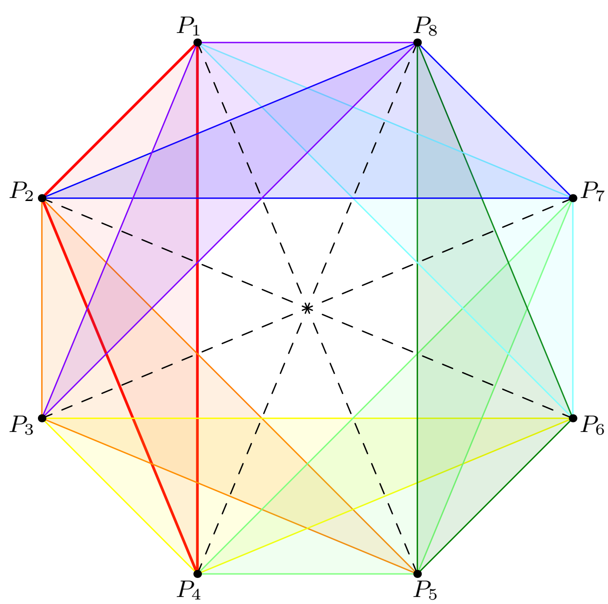
- If $\overline{AB}$ is an edge of the cube, then $AB = s$, so one possibility is $s_1 = \sqrt{6}$.
- If $\overline{AB}$ is a face diagonal of the cube, then $AB = s\sqrt{2}$, so another possibility is $s_2 = \sqrt{3}$.
- If $\overline{AB}$ is a space diagonal of the cube, then $AB = s\sqrt{3}$, so the last possibility is $s_3 = \sqrt{2}$.

The answer is then $s_1^3 s_2^3 s_3^3 = (s_1 s_2 s_3)^3 = 6^3 = \mathbf{216}$.

Problem 2. Eight students attend a Harper Valley ARML practice. At the end of the practice, they decide to take selfies to celebrate the event. Each selfie will have either two or three students in the picture. Compute the minimum number of selfies so that each pair of the eight students appears in exactly one selfie.

Solution 2. The answer is 12. To give an example in which 12 selfies is possible, consider regular octagon $P_1P_2P_3P_4P_5P_6P_7P_8$. Each vertex of the octagon represents a student and each of the diagonals and sides of the octagon represents a pair of students. Construct eight triangles $P_1P_2P_4$, $P_2P_3P_5$, $P_3P_4P_6$, $\ldots$, $P_8P_1P_3$. Each of the segments in the forms of $\overline{P_iP_{i+1}}$, $\overline{P_iP_{i+2}}$, $\overline{P_iP_{i+3}}$ appears exactly once in these eight triangles. Taking 8 three-person selfies (namely $\{P_1, P_2, P_4\}$, $\{P_2, P_3, P_5\}$, $\ldots$, $\{P_8, P_1, P_3\}$) and 4 two-person selfies (namely $\{P_1, P_5\}$, $\{P_2, P_6\}$, $\{P_3, P_7\}$, $\{P_4, P_8\}$) gives a total of 12 selfies, completing the desired task.

A diagram of this construction is shown below. Each of the eight triangles is a different color, and each of the two-person selfies is represented by a dotted diameter.



It remains to show fewer than 12 selfies is impossible. Assume that the students took x three-person selfies and y two-person selfies. Each three-person selfie counts 3 pairs of student appearances (in a selfie), and each two-person selfie counts 1 pair of student appearances (in a selfie). Together, these selfies count $3x + y$ pairs of student appearances. There are $\binom{8}{2} = 28$ pairs of student appearances. Hence $3x + y = 28$. The number of selfies is $x + y = 28 - 2x$, so it is enough to show that $x \le 8$.

Assume for contradiction there are $x \ge 9$ three-person selfies; then there are at least $3 \cdot 9 = 27$ (individual) student appearances on these selfies. Because there are 8 students, some student s_1 had at least $\lceil 27/8 \rceil$ appearances; that is, s_1 appeared in at least 4 of these three-person selfies. There are $2 \cdot 4 = 8$ (individual) student appearances other than s_1 on these 4 selfies. Because there are only 7 students besides s_1, some other student s_2 had at least $\lceil 8/7 \rceil$ (individual) appearances on these 4 selfies; that is, s_2 appeared (with s_1) in at least 2 of these 4 three-person selfies, violating the condition that each pair of the students appears in exactly one selfie. Thus the answer is **12**.

Problem 3. Compute the least positive value of t such that

$$\operatorname{Arcsin}(\sin(t)),\ \operatorname{Arccos}(\cos(t)),\ \operatorname{Arctan}(\tan(t))$$

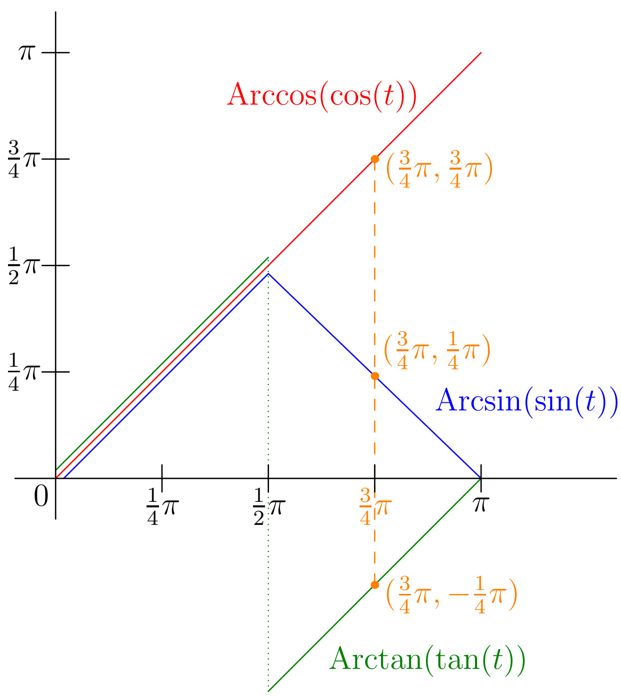
form (in some order) a three-term arithmetic progression with a nonzero common difference.

Solution 3. For $0 \le t < \pi/2$, all three values are t, so the desired t does not lie in this interval.

For $\pi/2 < t < \pi$,

$$\begin{aligned} \operatorname{Arcsin}(\sin(t)) &= \pi - t \in (0, \pi/2) \\ \operatorname{Arccos}(\cos(t)) &= t \qquad\ \in (\pi/2, \pi) \\ \operatorname{Arctan}(\tan(t)) &= t - \pi \in (-\pi/2, 0)\,. \end{aligned}$$

A graph of all three functions is shown below.



Thus if the three numbers are to form an arithmetic progression, they should satisfy

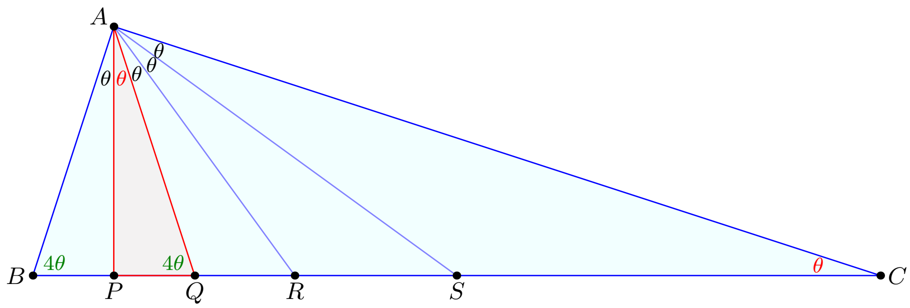
$$t - \pi < \pi - t < t.$$

The three numbers will be in arithmetic progression if and only if $t + (t - \pi) = 2(\pi - t)$, which implies $t = \boldsymbol{\frac{3\pi}{4}}$. Note that if $t = \frac{3\pi}{4}$, the arithmetic progression is $-\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$, as required.

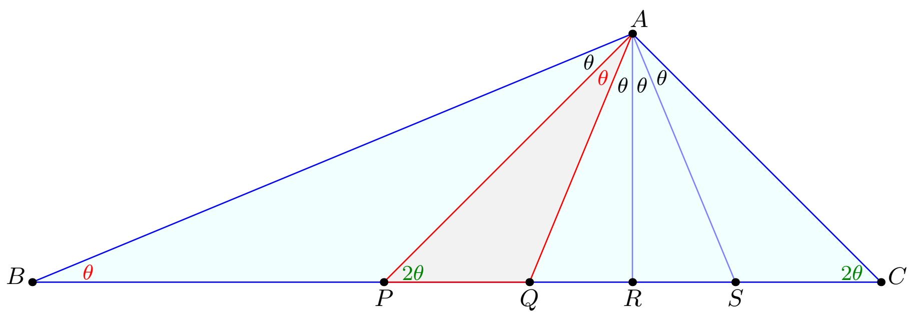
Problem 4. In non-right triangle ABC, distinct points P, Q, R, and S lie on $\overline{BC}$ in that order such that $\angle BAP \cong \angle PAQ \cong \angle QAR \cong \angle RAS \cong \angle SAC$. Given that the angles of $\triangle ABC$ are congruent to the angles of $\triangle APQ$ in some order of correspondence, compute m$\angle B$ in degrees.

Solution 4. Let $\theta = \frac{1}{5}\mathrm{m}\angle A$. Because $\mathrm{m}\angle PAQ = \theta < 5\theta = \mathrm{m}\angle A$, it follows that either $\mathrm{m}\angle B = \theta$ or $\mathrm{m}\angle C = \theta$. Thus there are two cases to consider.

If $\mathrm{m}\angle C = \theta$, then it follows that $\mathrm{m}\angle AQP = \mathrm{m}\angle QAC + \mathrm{m}\angle ACB = 4\theta$, and hence $\mathrm{m}\angle B = 4\theta$. So $\triangle ABC$ has angles of measures 5θ, 4θ, θ, and thus $\theta = 18°$. However, this implies $\mathrm{m}\angle A = 5\theta = 90°$, which is not the case.



If instead $\mathrm{m}\angle B = \theta$, then it follows that $\mathrm{m}\angle APQ = \mathrm{m}\angle BAP + \mathrm{m}\angle ABP = 2\theta$, and hence $\mathrm{m}\angle C = 2\theta$. So $\triangle ABC$ has angles of measures 5θ, 2θ, θ, and thus $\theta = 22.5°$. Hence $\mathrm{m}\angle B = \theta = \mathbf{22.5°}$.



Problem 5. Consider the system of equations

$$\begin{aligned}\log_4 x + \log_8(yz) &= 2\\ \log_4 y + \log_8(xz) &= 4\\ \log_4 z + \log_8(xy) &= 5.\end{aligned}$$

Given that xyz can be expressed in the form 2^k, compute k.

Solution 5. Note that for $n > 0$, $\log_4 n = \log_{64} n^3$ and $\log_8 n = \log_{64} n^2$. Adding together the three given equations and using both the preceding facts and properties of logarithms yields

$$\begin{aligned}&\log_4(xyz) + \log_8(x^2y^2z^2) = 11\\ \Longrightarrow\ &\log_{64}(xyz)^3 + \log_{64}(xyz)^4 = 11\\ \Longrightarrow\ &\log_{64}(xyz)^7 = 11\\ \Longrightarrow\ &7\log_{64}(xyz) = 11.\end{aligned}$$

The last equation is equivalent to $xyz = 64^{11/7} = 2^{66/7}$, hence the desired value of k is $\mathbf{\frac{66}{7}}$.

Problem 6. A complex number z is selected uniformly at random such that $|z| = 1$. Compute the probability that z and z^{2019} both lie in Quadrant II in the complex plane.

Solution 6. For convenience, let $\alpha = \pi/4038$. Denote by

$$0 \le \theta < 2\pi = 8076\alpha$$

the complex argument of z, selected uniformly at random from the interval $[0, 2\pi)$. Then z itself lies in Quadrant II if and only if

$$2019\alpha = \frac{\pi}{2} < \theta < \pi = 4038\alpha.$$

On the other hand, z^{2019} has argument 2019θ, and hence it lies in Quadrant II if and only if there is some integer k with

$$\begin{aligned} &\frac{\pi}{2} + 2k\pi < 2019\theta < \pi + 2k\pi \\ \iff &(4k+1) \cdot \frac{\pi}{2} < 2019\theta < (4k+2) \cdot \frac{\pi}{2} \\ \iff &(4k+1)\alpha < \theta < (4k+2)\alpha. \end{aligned}$$

Because it is also true that $2019\alpha < \theta < 4038\alpha$, the set of θ that satisfies the conditions of the problem is the union of intervals:

$$(2021\alpha, 2022\alpha) \cup (2025\alpha, 2026\alpha) \cup \cdots \cup (4037\alpha, 4038\alpha).$$

There are 505 such intervals, the j^{th} interval consisting of $(4j + 2017)\alpha < \theta < (4j + 2018)\alpha$. Each interval has length α, so the sum of the intervals has length 505α. Thus the final answer is

$$\frac{505\alpha}{2\pi} = \frac{505}{2 \cdot 4038} = \frac{\mathbf{505}}{\mathbf{8076}}.$$

Problem 7. Compute the least positive integer n such that the sum of the digits of n is five times the sum of the digits of $(n + 2019)$.

Solution 7. Let $S(n)$ denote the sum of the digits of n, so that solving the problem is equivalent to solving $S(n) = 5S(n + 2019)$. Using the fact that $S(n) \equiv n \pmod 9$ for all n, it follows that

$$\begin{aligned} n &\equiv 5(n + 2019) \equiv 5(n + 3) \pmod 9 \\ 4n &\equiv -15 \pmod 9 \\ n &\equiv 3 \pmod 9. \end{aligned}$$

Then $S(n + 2019) \equiv 6 \pmod 9$. In particular, $S(n + 2019) \ge 6$ and $S(n) \ge 5 \cdot 6 = 30$. The latter inequality implies $n \ge 3999$, which then gives $n + 2019 \ge 6018$. Thus if $n + 2019$ were a four-digit number, then $S(n + 2019) \ge 7$. Moreover, $S(n + 2019)$ can only be 7, because otherwise, $S(n) = 5S(n + 2019) \ge 40$, which is impossible (if n has four digits, then $S(n)$ can be no greater than 36). So if $n + 2019$ were a four-digit number, then $S(n + 2019) = 7$ and $S(n) = 35$. But this would imply that the digits of n are 8, 9, 9, 9 in some order, contradicting the assumption that $n + 2019$ is a four-digit number. On the other hand, if $n + 2019$ were a five-digit number such that $S(n + 2019) \ge 6$, then the least such value of $n + 2019$ is 10005, and indeed, this works because it corresponds to $n = \mathbf{7986}$, the least possible value of n.

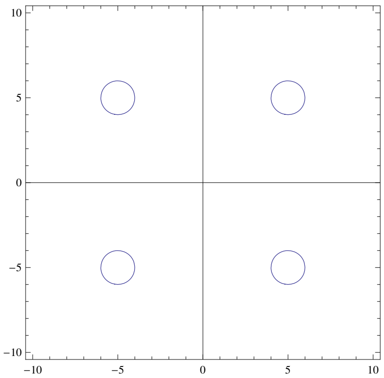
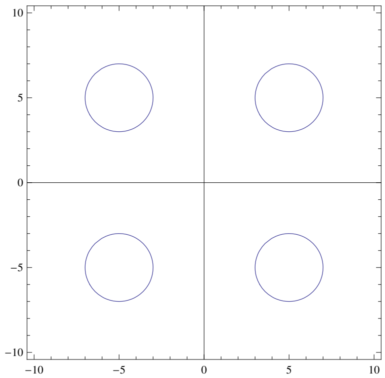
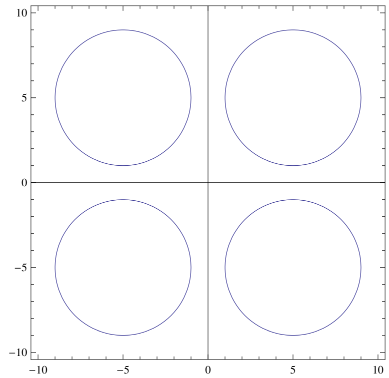
Problem 8. Compute the greatest real number K for which the graphs of

$$(|x| - 5)^2 + (|y| - 5)^2 = K \quad \text{and} \quad (x - 1)^2 + (y + 1)^2 = 37$$

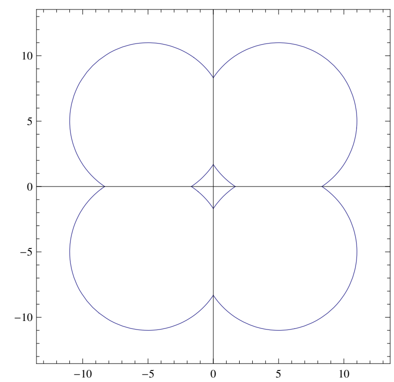
have exactly two intersection points.

Solution 8. The graph of the second equation is simply the circle of radius $\sqrt{37}$ centered at $(1,-1)$. The first graph is more interesting, and its behavior depends on K.

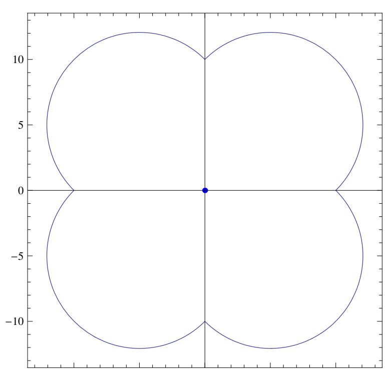
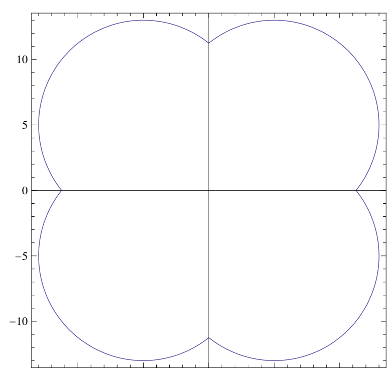
- For small values of K, the first equation determines a set of four circles of radius $\sqrt{K}$ with centers at $(5,5),(5,-5),(-5,5)$, and $(-5,-5)$. Shown below are versions with $K=1$, $K=4$, and $K=16$.

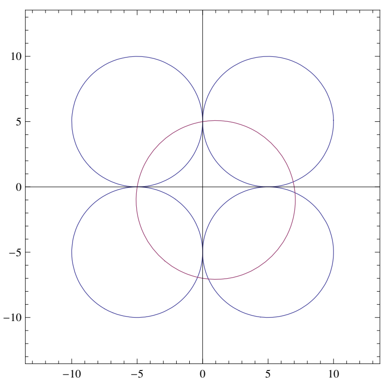
- However, when $K>25$, the graph no longer consists of four circles! As an example, for $K=36$, the value $x=5$ gives $(|y|-5)^2=36$; hence $|y|=-1$ or $|y|=6$. The first option is impossible; the graph ends up "losing" the portions of the upper-right circle that would cross the x- or y-axes compared to the graph for $(x-5)^2+(y-5)^2=36$. The graph for $K=36$ is shown below.



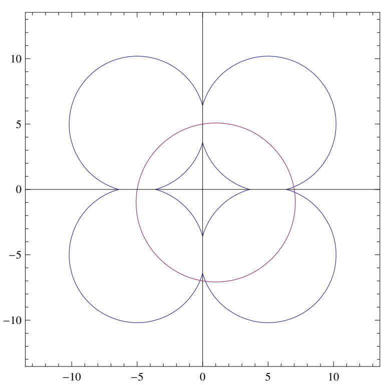
- As K continues to increase, the "interior" part of the curve continues to shrink, until at $K=50$, it simply comprises the origin, and for $K>50$, it does not exist. As examples, the graphs with $K=50$ and $K=64$ are shown below.

Overlay the graph of the circle of radius $\sqrt{37}$ centered at $(1, -1)$ with the given graphs. When $K = 25$, this looks like the following graph.



Note that the two graphs intersect at $(0, 5)$ and $(-5, 0)$, as well as four more points (two points near the positive x-axis and two points near the negative y-axis). When K is slightly greater than 25, this drops to four intersection points. The graph for $K = 27$ is shown below.



Thus for the greatest K for which there are exactly two intersection points, those two intersection points should be along the positive x- and negative y-axes. If the intersection point on the positive x-axis is at $(h, 0)$, then $(h-1)^2 + (0+1)^2 = 37$ and $(h-5)^2 + (0-5)^2 = K$. Thus $h = 7$ and $K = \mathbf{29}$.

Problem 9. To *morph* a sequence means to replace two terms a and b with $a+1$ and $b-1$ if and only if $a+1 < b-1$, and such an operation is referred to as a morph. Compute the least number of morphs needed to transform the sequence $1^2, 2^2, 3^2, \ldots, 10^2$ into an arithmetic progression.

Solution 9. Call the original sequence of ten squares $T = (1^2, 2^2, \ldots, 10^2)$. A *morphed sequence* is one that can be obtained by morphing T a finite number of times.

This solution is divided into three steps. In the first step, a characterization of the possible final morphed sequences is given. In the second step, a lower bound on the number of steps is given, and in the third step, it is shown that this bound can be achieved.

Step 1. Note the following.

- The sum of the elements of T is $1^2 + 2^2 + \cdots + 10^2 = 385$, and morphs are sum-preserving. So any morphed sequence has sum 385 and a mean of 38.5.
- The sequence T has positive integer terms, and morphs preserve this property. Thus any morphed sequence has positive integer terms.
- The sequence T is strictly increasing, and morphs preserve this property. Thus any morphed sequence is strictly increasing.

Now if the morphed sequence is an arithmetic progression, it follows from the above three observations that it must have the form

$$(38.5 - 4.5d,\ 38.5 - 3.5d,\ \ldots,\ 38.5 + 4.5d)$$

where d is an odd positive integer satisfying $38.5 - 4.5d > 0$. Therefore the only possible values of d are 7, 5, 3, 1; thus there are at most four possibilities for the morphed sequence, shown in the table below. Denote these four sequences by A, B, C, D.

	T	1	4	9	16	25	36	49	64	81	100
$d = 7:$	A	7	14	21	28	35	42	49	56	63	70
$d = 5:$	B	16	21	26	31	36	41	46	51	56	61
$d = 3:$	C	25	28	31	34	37	40	43	46	49	52
$d = 1:$	D	34	35	36	37	38	39	40	41	42	43

Step 2. Given any two sequences $X = (x_1, \ldots, x_{10})$ and $Y = (y_1, \ldots, y_{10})$ with $\sum_{i=1}^{10} x_i = \sum_{i=1}^{10} y_i = 385$, define the *taxicab distance*

$$\rho(X, Y) = \sum_{i=1}^{10} |x_i - y_i|.$$

Observe that if X' is a morph of X, then $\rho(X', Y) \geq \rho(X, Y) - 2$. Therefore the number of morphs required to transform T into some sequence Z is at least $\frac{1}{2}\rho(T, Z)$. Now

$$\frac{1}{2}\rho(T, A) = \frac{1}{2}\sum_{i=1}^{10} \left|i^2 - 7i\right| = 56$$

and also $\rho(T, A) < \min\left(\rho(T, B), \rho(T, C), \rho(T, D)\right)$. Thus at least 56 morphs are needed to obtain sequence A (and more morphs would be required to obtain any of sequences B, C, or D).

Step 3. To conclude, it remains to verify that one can make 56 morphs and arrive from T to A. One of many possible constructions is given below.

T	1	4	9	16	25	36	49	64	81	100
6 morphs	1	4	9	16	25	42	49	58	81	100
2 morphs	1	4	9	16	27	42	49	56	81	100
8 morphs	1	4	9	16	35	42	49	56	73	100
10 morphs	1	4	9	26	35	42	49	56	63	100
2 morphs	1	4	9	28	35	42	49	56	63	98
12 morphs	1	4	21	28	35	42	49	56	63	86
10 morphs	1	14	21	28	35	42	49	56	63	76
6 morphs	7	14	21	28	35	42	49	56	63	70

Therefore the least number of morphs needed to transform T into an arithmetic progression is **56**.

Remark: For step 3, one may prove more generally that any sequence of 56 morphs works as long as both of the following conditions hold:

- each morph increases one of the first six elements and decreases one of the last three elements, and
- at all times, the i^{th} term is at most $7i$ for $i \leq 6$, and at least $7i$ for $i \geq 8$.

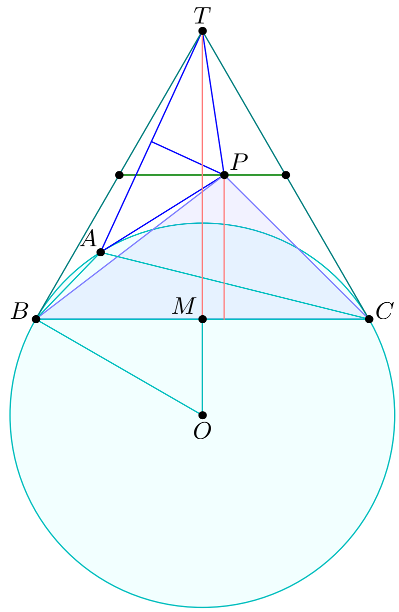
Problem 10. Triangle ABC is inscribed in circle ω. The tangents to ω at B and C meet at point T. The tangent to ω at A intersects the perpendicular bisector of $\overline{AT}$ at point P. Given that $AB = 14$, $AC = 30$, and $BC = 40$, compute $[PBC]$.

Solution 10. To begin, denote by R the radius of ω. The semiperimeter of triangle ABC is 42, and then applying Heron's formula yields

$$[ABC] = \frac{14 \cdot 30 \cdot 40}{4R} = \sqrt{42 \cdot 28 \cdot 12 \cdot 2} = 168$$

from which it follows that $R = \dfrac{14 \cdot 30 \cdot 40}{4 \cdot 168} = 25$.

Now consider the point circle with radius zero centered at T in tandem with the circle ω. Because $PA = PT$, it follows that P lies on the radical axis of these circles. Moreover, the midpoints of $\overline{TB}$ and $\overline{TC}$ lie on this radical axis as well. Thus P lies on the midline of $\triangle TBC$ that is parallel to $\overline{BC}$.



To finish, let O denote the center of ω and M the midpoint of $\overline{BC}$. By considering right triangle TBO with altitude $\overline{BM}$, it follows that $MT \cdot MO = MB^2$, but also $MO = \sqrt{OB^2 - MB^2} = \sqrt{25^2 - 20^2} = 15$, so

$$MT = \frac{MB^2}{MO} = \frac{400}{15} = \frac{80}{3}.$$

Thus the distance from P to $\overline{BC}$ is $\frac{1}{2}MT = \frac{40}{3}$. Finally,

$$[PBC] = \frac{1}{2} \cdot \frac{40}{3} \cdot BC = \mathbf{\frac{800}{3}}.$$