CT ARML Team, 2024

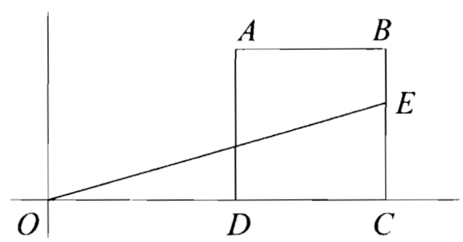
Team Selection Test 2

1. Let A and B be digits, $A > B$, and suppose that $\underline{A}\,\underline{B}\,\underline{A}_{10} - \underline{B}\,\underline{A}\,\underline{B}_{10}$ is divisible by exactly three distinct prime numbers. Compute the largest possible value of B. (Here, $\underline{A}\,\underline{B}\,\underline{C}_{10}$ represents the base-10 number whose digits are A, B, and C, in that order.)

2. Point B lies at the origin, $A = (2, 8)$, and $C = (4, 2)$. Suppose that $D = (a, b)$, where $a > 0$, $b > 0$, and the midpoints of the sides of quadrilateral $ABCD$ form a square. Find $2a + 3b$.

3. The sum of k consecutive integers is 2024. (The integers are <u>not</u> necessarily positive.) What is the largest possible value of k?

4. In the diagram below, square $ABCD$ is divided into two regions of equal area by $\overline{OE}$. The lengths OD and AD are integers, and $\frac{CE}{BE} = 1000$. Compute the smallest possible value of AD.



5. The roots of the equation $ax^2 + bx + c = 0$ are irrational, and, correct to ten decimal places, their values are 0.8430703308 and -0.5930703308. It is further given that a, b, and c are integers, $\gcd(a, b, c) = 1$, $a > 0$, $|b| \leq 10$ and $|c| \leq 10$. Then $a = p$, $b = (-1)^m \cdot q$, and $c = (-1)^n \cdot r$, where p, q, r are positive integers, m is 0 or 1, and n is 0 or 1. Find $p + q + r + m + n$.

6. Suppose that x and y are integers such that $y^2 + 5x^2y^2 = 30x^2 + 909$. Find $x^2 + y^2$.

7. Let $f(x) = \sqrt{x + 2} + c$ and let S be the set of values of c such that the graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect at two distinct points. Then the greatest lower bound of S is $(-1)^n \frac{p}{q}$, where p and q are relatively prime positive integers and n is 0 or 1. Find $p + q + n$.

8. How many permutations $(a_1, a_2, \ldots, a_{10})$ of the numbers $1, 2, 3, \cdots, 10$ satisfy the equation $\left|a_1 - 1\right| + \left|a_2 - 2\right| + \ldots + \left|a_{10} - 10\right| = 4$?

9. Let $a_1, a_2, \ldots$ be a sequence defined by $a_1 = 1$ and, for $n \geq 1$,

$$a_{n+1} = \sqrt{a_n^2 - 2a_n + 3} + 1$$

Compute a_{513}.

10. Find the sum of all real numbers x such that $1 \leq x \leq 99$ and $\left\{x^2\right\} = \{x\}^2$.
(Here, $\{x\}$ denotes the fractional part of x.)

Team Selection Test 2 Answers

1. 7
2. 18
3. 4048
4. 999
5. 9
6. 53
7. 14
8. 52
9. 33
10. 641999

CT ARML Team, 2024

Team Selection Test 2 Solutions

1. Let A and B be digits, $A > B$, and suppose that $\underline{A}\,\underline{B}\,\underline{A}_{10}$ $\underline{B}\,\underline{A}\,\underline{B}_{10}$ is divisible by exactly three distinct prime numbers. Compute the largest possible value of B.
   (Here, $\underline{A}\,\underline{B}\,\underline{C}_{10}$ represents the base-10 number whose digits are A, B, and C, in that order.)
   [Answer: 7]

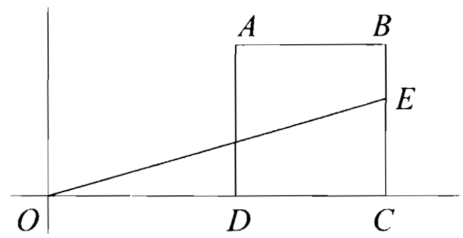
   Solution:
   $\underline{A}\,\underline{B}\,\underline{A}_{10} - \underline{B}\,\underline{A}\,\underline{B}_{10}$ = (A - B) * (100 - 10 + 1) = (A - B) * 91 = (A - B) * 13 * 7 which can have exactly 3 prime divisors only if (A - B) = (p ^ a) * (7 ^ b) * (13^c), where p is a prime not equal to 7 or 13. B is largest if A = 9 and (A - B) is smallest, and so p = 2, a = 1, b = c = 0. Hence A-B = 2, A = 9, B = 7

   [Answer: 7]

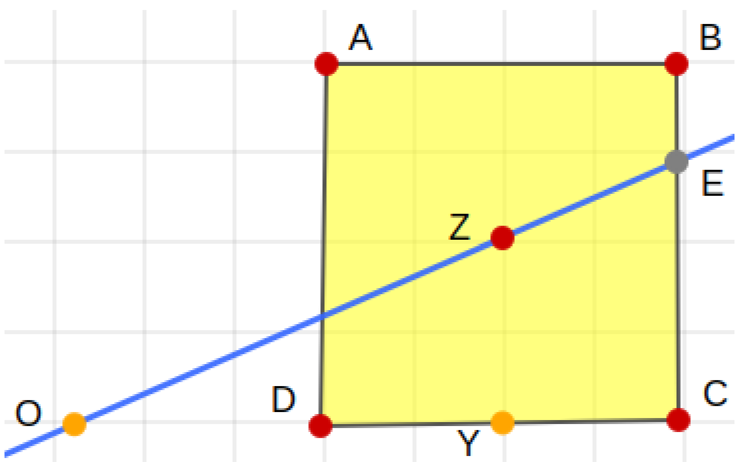
2. Point B lies at the origin, $A = (2, 8)$, and $C = (4, 2)$. Suppose that $D = (a, b)$, where $a > 0,\ b > 0$, and the midpoints of the sides of quadrilateral $ABCD$ form a square. Find $2a + 3b$.
   [Answer: 18]
   Solution:
   The coordinates of midpoints of AB and BC are (1, 4) and (2, 1) respectively. The vector from the midpoint of AB to the midpoint of BC is <-1,3>. Thus the vector from the midpoint of AB to the midpoint of AD is <3,1> since they are perpendicular to each other. Thus the midpoint of AD is at (4,5), and the coordinate of point D is (6,2).
   $$2a + 3b = 18$$

3. The sum of k consecutive integers is 2024. (The integers are <u>not</u> necessarily positive.) What is the largest possible value of k?
   [Answer: 4048]
   Solution:
   To maximize the value of k, we want to have most numbers canceled in the series of consecutive integers and leave the last number 2024.
   Or we can center around the two numbers 0, 1, with each pair of $0 - n,\ 1 + n$ that adds 1 to the sum.
   So the answer is 4048.

4. In the diagram below, square $ABCD$ is divided into two regions of equal area by $\overline{OE}$. The lengths OD and AD are integers, and $\frac{CE}{BE} = 1000$. Compute the smallest possible value of AD.



[Answer: 999]
Solution:
Let OD = y and AD = AB = BC = CD = x, both positive integers.
Since OE divides ABCD into equal areas it goes through the center of the square.



Let the center be Z and the foot of the perpendicular from Z to ODC be Y. Then we have
CE = (1000/1001) * x
and ZY/OY = EC/OC
or (x/2)/(y + (x/2)) = (1000/1001)x/(y + x)
or, simplifying, x = 999 y
So the smallest positive value for x = 999
[Answer: 999]

5. The roots of the equation $ax^2 + bx + c = 0$ are irrational, and, correct to ten decimal places, their values are 0.8430703308 and −0.5930703308. It is further given that a, b, and c are integers, $\gcd(a, b, c) = 1$, $a > 0$, $|b| \leq 10$ and $|c| \leq 10$. Then $a = p$,

$b = (-1)^{m} \cdot q$, and $c = (-1)^{n} \cdot r$, where p, q, r are positive integers, m is 0 or 1, and n is 0 or 1. Find $p + q + r + m + n$.
[Answer: 9]
Solution:
By Vieta, -(b/a) = 0.25 +- 10^{-10}, hence b is negative. Similarly c is also negative (Since a > 0)
(¼) - 10^{-10} <= (-b)/a <= (¼) + 10^{-10}
Or (-b)/((¼) + 10^{-10}) <= a <= (-b)/((¼) - 10^{-10})
Or 4*(-b) / (1 + 4E-10) <= a <= 4*(-b)/(1 - 4E-10)
We are trying to show from here that a = 4*(-b).
We will deal with the left inequality using (1 + x) (1-x) = 1 - x^2 < 1 for 0< x <1 which implies 1/(1+x) > (1-x). Applying this on the left inequality with x = 4E-10 (< 1)
4*(-b) * (1 - 4E-10) < a
For the right inequality we use 1/(1-x) < 1 + 2x for 0 < x < 0.5 (which is gotten by expanding out (1-x)(1+2x))
So combining everything, we get
4*(-b) * (1 - 4E-10) < a < 4*(-b) * (1 + 8E-10)
Thus | a - 4*(-b)| < 4 * (-b) * 1.2E-9
Since (-b) < 10, we get | a - 4*(-b)| < 1. Since both are integers, we get a = 4*(-b)

For the constant term, (product of roots by vieta), by long division upto 5 decimal points we get (-c)/a = 0.499999 +- 10^{-5}
Using the same inequalities, we get
So 0.5 - 2E-5 < 0.499999 - 10^{-5} < (-c)/a < 0.499999 + 10^{-5} < 0.5 + 2E-5
and with similar manipulation as above,
2(-c) * (1 - 2E-5) < a < 2(-c) * (1 + 4E-5)
or |a - 2(-c)| < 2(-c)*6E-5, and using -c < 10, we get as above a = 2(-c)

Using the gcd condition, we get b = -1, a = 4, c = -2, the roots are indeed irrational and as calculated.
Answer = 4 + 1+1 + 1+2 = 9

6. Suppose that x and y are integers such that $y^2 + 5x^2y^2 = 30x^2 + 909$. Find $x^2 + y^2$.
[Answer: 53]
Solution:
Use SFFT,

$$y^2 + 5x^2y^2 - 30x^2 = 909$$

$$y^2\left(1 + 5x^2\right) - 6\left(1 + 5x^2\right) = 903$$
$$\left(y^2 - 6\right)\left(5x^2 + 1\right) = 903 = 3\times7\times43$$

The only solution that satisfies the condition that x and y are integers is $x^2 = 4,\ y^2 = 49.$ So the answer is 53.

7. Let $f(x) = \sqrt{x + 2} + c$ and let S be the set of values of c such that the graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect at two distinct points. Then the greatest lower bound of S is $(-1)^n\frac{p}{q}$, where p and q are relatively prime positive integers and n is 0 or 1. Find $p + q + n$.
[Answer: 14]
Solution:

$y = f(x)$ and $y = f^{-1}(x)$ are always going to intersect on the line y = x. Also, if f and f^{-1} are differentiable, the respective slopes of their tangents (derivatives) at the extrema of S are going to be = 1. (I.e. for the curves that correspond to the extrema of S are tangents to one another and their derivatives at the point of tangency are both 1).

We will use a parameterization where the value of the derivative is the parameter (s). In this problem, since the first equation has a square root, we are only taking the positive square r

$f^{(1)}(x) = 1/(2\,sqrt(x + 2)) = s$ and y = (1/2s) + c and x = (1/2s)^2 - 2
At s = 1, y = ½ + C, x = - 7/4

Since we know that y = x, this implies C = - 9/4
We can check that these relations (y = x and s = 1) hold for the inverse curve. Hence the answer = 1 + 9 + 4 = 14

8. How many permutations $(a_1, a_2, \ldots, a_{10})$ of the numbers $1, 2, 3, \cdots, 10$ satisfy the equation $\left|a_1 - 1\right| + \left|a_2 - 2\right| + \ldots + \left|a_{10} - 10\right| = 4$?
Solution:
Consider the LHS of the equation, each term is a positive integer, and so constitute a partition of 4. The partitions of 4 are
4 = 4 ……………………… E
= 3 + 1 ………………… D

= 2 + 1 + 1 ……………….. C
= 2 + 2 ………………….. B
= 1 + 1 + 1 + 1 …………… A

E is ruled out, as it is not possible for only one item to map to something other than itself in a permutation.
D is ruled out, as if there are only 2 non-zero entries, |a_i - i| and |a_j - j|, the permutation necessarily is (i,j), and then |a_i - i| = |a_j - j| = |i-j|, and cannot be unequal.
C can only come from a 3-cycle (i,jk) where i, j, k are a consecutive triple.
B can only come from a swap (i, i+2)
A can only come from a combination of swaps: (i,i+1)(j,j+1). Since, by a cycle decomposition, if any cycle is of length > 2, it will have some |a_i - i| > 1.

(In the above, I am using the cycle notation for writing permutations)

Case C: Pick 3 successive numbers and do a cyclic permutation. 8 ways to pick 3 successive numbers ((1,2,3), (2,3,4), … (8,9,10)) and for each 2 cyclic permutations, hence 16 permutations

Case B: 8 cases ((1,3), (2,4), (3,5), …, (8,10))

Case A: Number of successive swaps = 9. If this swap is not (1,2) or (9,10), then choosing the next swap can be done in 10 - 2 - 2 = 6 ways. In case you pick (1,2) or (9,10), the next swap can be done in 10-2-1 = 7 ways. There is a overcounting by a factor of 2 since one can swap the swaps. So total number of permutations = (7 * 6 + 2 * 7)/2 = (8*7)/2 = 4 * 7 = 28 ways

Total = 28 + 8 + 16 = 4 * (7 + 2 + 4) 4 * 13 = 52

[Answer: 52]

9. Let $a_1, a_2,$ ... be a sequence defined by $a_1 = 1$ and, for $n \geq 1$,

$$a_{n+1} = \sqrt{a_n^{\ 2} - 2a_n + 3} + 1$$

Compute a_{513}.
[Answer: 33]
Solution:

Rewrite the given equation

$$a_{n+1} - 1 = \sqrt{a_n^2 - 2a_n + 1 + 2} = \sqrt{(a_n - 1)^2 + 2}$$

$$(a_{n+1} - 1)^2 = (a_n - 1)^2 + 2$$

Let $b_n = (a_n - 1)^2$, then $b_{n+1} = b_n + 2$.
Since $a_1 = 1,\ b_n = 0$, we can deduce $b_n = 2(n - 1)$. Thus

$$b_{513} = 1024$$

$$\left(a_{513} - 1\right)^2 = 1024$$

$$a_{513} = 33$$

10. Find the sum of all real numbers x such that $1 \le x \le 99$ and $\left\{x^2\right\} = \{x\}^2$.
(Here, $\{x\}$ denotes the fractional part of x.)
[Answer: 641999]
Solution:
For the real number x, the integral part is $\lfloor x \rfloor$and the fractional part is $\{x\}$

$$\left\{x^2\right\} = \{x\}^2$$

$$x^2 - \lfloor x^2 \rfloor = (x - \lfloor x \rfloor)^2 = x^2 - 2x\lfloor x \rfloor + \lfloor x \rfloor^2$$

$$\lfloor x^2 \rfloor = 2x\lfloor x \rfloor - \lfloor x \rfloor^2$$

Let $\lfloor x \rfloor = k$, the RHS $(2kx - k^2)$ must be an integer. Assume LHS = $k^2 + a$ with a being an integer. So

$$2kx - k^2 = k^2 + a$$

$$x = k + \frac{a}{2k},\ with\ 0 \le a < 2k$$

It is easy to prove that for all positive integer a in the range of $[0,\ 2k)$ the equation works.

$$\left\{x^2\right\} = \left\{(k + \tfrac{a}{2k})^2\right\} = \left\{k^2 + a + (\tfrac{a}{2k})^2\right\} = (\tfrac{a}{2k})^2 = \{x\}^2$$

So, the sum is

$$99 + \sum_{k=1}^{98}\left[k + \left(k + \frac{1}{2k}\right) + \left(k + \frac{2}{2k}\right) + \dots + \left(k + \frac{2k-1}{2k}\right)\right]$$

$$= 99 + \sum_{k=1}^{98} (2k^2 + \frac{1}{2k} \sum_{a=0}^{2k-1} a) = 99 + \sum_{k=1}^{98} (2k^2 + k - \frac{1}{2})$$

$$= 99 + 2 \times \frac{98 \times 99 \times 197}{6} + \frac{98 \times 99}{2} - 49 = 641899$$

4 Power Question 2019: Elizabeth's Escape

Instructions: The power question is worth 50 points; each part's point value is given in brackets next to the part. To receive full credit, the presentation must be legible, orderly, clear, and concise. If a problem says "list" or "compute," you need not justify your answer. If a problem says "determine," "find," or "show," then you must show your work or explain your reasoning to receive full credit, although such explanations do not have to be lengthy. If a problem says "justify" or "prove," then you must prove your answer rigorously. Even if not proved, earlier numbered items may be used in solutions to later numbered items, but not vice versa. Pages submitted for credit should be NUMBERED IN CONSECUTIVE ORDER AT THE TOP OF EACH PAGE in what your team considers to be proper sequential order. PLEASE WRITE ON ONLY ONE SIDE OF THE ANSWER PAPERS. Put the TEAM NUMBER (not the team name) on the cover sheet used as the first page of the papers submitted. Do not identify the team in any other way.

Elizabeth is in an "escape room" puzzle. She is in a room with one door which is locked at the start of the puzzle. The room contains n light switches, each of which is initially off. Each minute, she must flip exactly k different light switches (to "flip" a switch means to turn it on if it is currently off, and off if it is currently on). At the end of each minute, if all of the switches are on, then the door unlocks and Elizabeth escapes from the room.

Let $E(n,k)$ be the minimum number of minutes required for Elizabeth to escape, for positive integers n,k with $k \le n$. For example, $E(2,1)=2$ because Elizabeth cannot escape in one minute (there are two switches and one must be flipped every minute) but she can escape in two minutes (by flipping Switch 1 in the first minute and Switch 2 in the second minute). Define $E(n,k)=\infty$ if the puzzle is impossible to solve (that is, if it is impossible to have all switches on at the end of any minute).

For convenience, assume the n light switches are numbered 1 through n.

1. Compute the following.

- **a.** $E(6,1)$ [1 pt]
- **b.** $E(6,2)$ [1 pt]
- **c.** $E(7,3)$ [1 pt]
- **d.** $E(9,5)$ [1 pt]

2. Find the following in terms of n.

- **a.** $E(n,2)$ for positive even integers n [2 pts]
- **b.** $E(n,3)$ for values of n of the form $n=3a+2$ where a is a positive integer [2 pts]
- **c.** $E(n,n-2)$ for $n \ge 5$ [2 pts]

3. Find an integer value of k with $1<k<2019$ such that $E(2019,k)=\infty$. [3 pts]

4. **a.** Show that if $n+k$ is even and $\frac{n}{2}<k<n$, then $E(n,k)=3$. [3 pts]

b. Show that if n is even and k is odd, then $E(n,k)=E(n,n-k)$. [3 pts]

5. Find the following.

- **a.** $E(2020,1993)$ [3 pts]
- **b.** $E(2001,501)$ [3 pts]

6. **a.** Show that if n and k are both even and $k \le \frac{n}{2}$, then $E(n,k)=\lceil \frac{n}{k} \rceil$. [3 pts]

b. Prove that if k is odd and $k \le \frac{n}{2}$, then either $E(n,k)=\lceil \frac{n}{k} \rceil$ or $E(n,k)=\lceil \frac{n}{k} \rceil+1$. [4 pts]

7. Find all ordered pairs (n,k) for which $E(n,k)=3$. [3 pts]

One might guess that in most cases, $E(n,k) \approx \frac{n}{k}$. In light of this guess, define the *inefficiency* of the ordered pair (n,k), denoted $I(n,k)$, as

$$I(n,k) = E(n,k) - \frac{n}{k}$$

if $E(n,k) \neq \infty$. If $E(n,k) = \infty$, then by convention, $I(n,k)$ is undefined.

8. **a.** Compute $I(6,3)$. [1 pt]

b. Compute $I(5,3)$. [1 pt]

c. Find positive integers n and k for which $I(n,k) = \frac{15}{8}$. [2 pts]

d. Prove that for any integer $x > 2$, there exists an ordered pair (n,k) for which $I(n,k) > x$. [3 pts]

9. Let S be the set of values of $I(n,k)$ for all n,k for which $k < \frac{n}{2}$ and $I(n,k)$ is defined. Find the least upper bound of S. Prove that your answer is correct. [4 pts]

10. Find two distinct non-integral positive rational numbers that are not the inefficiency of any ordered pair. That is, find positive rational numbers q_1 and q_2 with $q_1 \neq q_2$ such that neither q_1 nor q_2 is an integer and such that neither q_1 nor q_2 is $I(n,k)$ for any integers n and k. Prove that your answers are correct. [4 pts]

5 Solutions to Power Question

First, notice that a light switch is on if it has been flipped an odd number of times, and off if it has been flipped an even number of times.

We use the notation $\{a_1, a_2, \ldots, a_k\}$ to denote the set of k switches flipped in any given minute.

1. **a.** $E(6,1) = 6$. Note that at least six minutes are required because exactly one switch is flipped each minute. By flipping all six switches (in any order) in the first six minutes, the door will open in six minutes.

b. $E(6,2) = 3$. The sequence $\{1,2\}$, $\{3,4\}$, $\{5,6\}$ will allow Elizabeth to escape the room in three minutes. It is not possible to escape the room in fewer than three minutes because every switch must be flipped, and that requires at least $\frac{6}{2} = 3$ minutes.

c. $E(7,3) = 3$. First, note that $E(7,3) \geq 3$, because after only two minutes, it is impossible to flip each switch at least once. It is possible to escape in three minutes with the sequence $\{1,2,3\}$, $\{1,4,5\}$, and $\{1,6,7\}$.

d. $E(9,5) = 3$. Notice that $E(9,5) \neq 1$ because each switch must be flipped at least once, and only five switches can be flipped in one minute. Notice also that $E(9,5) \neq 2$ because after two minutes, there have been 10 flips, but in order to escape the room, each switch must be flipped at least once, and this requires 9 of the 10 flips. However, the tenth flip of a switch returns one of the nine switches to the off position, so it is not possible for Elizabeth to escape in two minutes. In three minutes, however, Elizabeth can escape with the sequence $\{1,2,3,4,5\}$, $\{1,2,3,6,7\}$, $\{1,2,3,8,9\}$.

2. **a.** If n is even, then $E(n,2) = \frac{n}{2}$. This is the minimum number of minutes required to flip each switch at least once, and Elizabeth can clearly escape in $\frac{n}{2}$ minutes by flipping each switch *exactly* once.

b. If $n = 3a + 2$ $(a \geq 1)$, then $E(n,3) = a + 2$. The minimum number of minutes required to flip each switch once is $a + 1$, but as in Problem 1d, this leaves exactly one "extra flip", so some switch must be flipped exactly twice. However, in $a + 2 = \frac{n+4}{3}$ minutes, Elizabeth can escape by starting with the sequence $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$, and flipping each remaining switch exactly once.

c. If $n \geq 5$, then $E(n, n-2) = 3$. Note that Elizabeth cannot flip every switch in one minute, and after two minutes, some switch (in fact, many switches) must be flipped exactly twice. However, Elizabeth can escape in three minutes using the sequence $\{1,4,5,\ldots,n\}$, $\{2,4,5,\ldots,n\}$, $\{3,4,5,\ldots,n\}$.

3. The answer is that k can be any even number between 2 and 2018 inclusive. If Elizabeth escapes, each switch must have been flipped an odd number of times. Because there are 2019 switches, the total number of flips must be odd. However, if an even number of flips are performed each minute, then the total number of flips cannot be odd. Therefore the puzzle is impossible, and $E(2019, k) = \infty$.

Alternate Solution: As in Problem 2a, consider the number of lights that are on after each minute. Flipping an even number of switches either leaves this number unchanged, or increases or decreases it by an even number. Because the puzzle starts with 0 lights on, and 2019 is odd, it is impossible to have 2019 lights on at the end of any minute. Thus $E(2019, k) = \infty$.

4. **a.** First, because $k < n$, not all switches can be flipped in one minute, and so $E(n,k) \neq 1$. Second, because $k > \frac{n}{2}$, some switches must be flipped twice in the first two minutes, and so $E(n,k) \neq 2$.

Here is a strategy by which Elizabeth can escape in three minutes. Note that because $n + k$ is even, it follows that $3k - n = (k + n) + 2(k - n)$ is also even. Then let $3k - n = 2b$. Note that $b > 0$ (because $k > \frac{n}{2}$), and $b < k$ (because $k < n$).

Elizabeth's strategy is to flip switches 1 through b three times each, and the remaining $n - b$ switches once each. This is possible because $n - b = (3k - 2b) - b = 3(k - b)$ is divisible by 3, and $b + \frac{n-b}{3} = b + (k - b) = k$. Therefore, each minute, Elizabeth can flip switches 1 through b, and one-third of the $3(k - b)$ switches from $b + 1$ through n. This allows her to escape in three minutes, as desired.

b. Because n is even, and because each switch must be flipped an odd number of times in order to escape, the total number of flips is even. Because k must be odd, $E(n,k)$ must be even. To show this, consider the case where $E(n,k)$ is odd. If $E(n,k)$ is odd, then an odd number of flips happen an odd number of times, resulting in an odd number of total flips. This is a contradiction because n is even.

Call a switch "non-flipped" in any given minute if it is not among the switches flipped in that minute. Because $E(n,k)$ (i.e., the total number of minutes) is even, and each switch is flipped an odd number of times, each switch must also be non-flipped an odd number of times. Therefore any sequence of flips that solves the "(n,k) puzzle" can be made into a sequence of flips that solves the "$(n,n-k)$" puzzle by interchanging flips and non-flips. These sequences last for the same number of minutes, and therefore $E(n,k) = E(n,n-k)$.

5. **a.** $E(2020,1993) = 76$. By the result of Problem 4, conclude that $E(2020,1993) = E(2020,27)$. Compute the latter instead. Because $\frac{2020}{27} > 74$, it will require at least 75 minutes to flip each switch once. Furthermore, $E(2020,27) \geq 76$ because the solution to Problem 4 implies that $E(2020,27)$ is even.

To solve the puzzle in exactly 76 minutes, use the following strategy. For the first 33 minutes, flip switch 1, along with the first 26 switches that have not yet been flipped. The end result is that lights 1 through $26 \cdot 33 + 1 = 859$ are on, and the remaining 1161 lights are off. Note that $1161 = 27 \cdot 43$, so it takes 43 minutes to flip each remaining switch exactly once, for a total of 76 minutes, as desired.

b. $E(2001,501) = 5$. First, note that three minutes is not enough time to flip each switch once. In four minutes, Elizabeth can flip each switch once, but has three flips left over. Because there are an odd number of leftover flips to distribute among the 2001 switches, some switch must get an odd number of leftover flips, and thus an even number of total flips. Thus $E(2001,501) > 4$.

To solve the puzzle in five minutes, Elizabeth can flip the following sets of switches:

- in the first minute, $\{1,2,3,\ldots,501\}$;
- in the second minute, $\{1,2,3,\ldots,102\}$ and $\{502,503,504,\ldots,900\}$;
- in the third minute, $\{1,2,3,\ldots,102\}$ and $\{901,902,903,\ldots,1299\}$;
- in the fourth minute, $\{1,2,3,\ldots,100\}$ and $\{1300,1301,1302,\ldots,1700\}$;
- in the fifth minute, $\{1,2,3,\ldots,100\}$ and $\{1701,1702,1703,\ldots,2001\}$.

This results in switches $1,2,3,\ldots,100$ being flipped five times, switches 101 and 102 being flipped three times, and the remaining switches being flipped exactly once, so that all the lights are on at the end of the fifth minute.

6. **a.** First, if k divides n, then $E(n,k) = \frac{n}{k} = \lceil \frac{n}{k} \rceil$. Assume then that k does not divide n. Then let $r = \lceil \frac{n}{k} \rceil$, which implies $(r-1)k < n < rk$.

Because $(r-1)k < n$, it follows that $r-1$ minutes are not enough to flip each switch once, so $E(n,k) \geq r$.

Because n and k are even, it follows that rk and $rk-n$ are also even. Then $rk - n = 2b$ for some integer $b \geq 1$, and note that $b < k$ because $rk - k < n$. Then the following strategy turns all of the lights on in r minutes.

- In each of the first three minutes, flip switches 1 through b, along with the next $k-b$ switches that have not yet been flipped. This leaves $b + 3(k-b)$ lights on, and the rest off. Note that $b + 3(k-b) = 3k - 2b = 3k - (rk-n) = n - k(r-3)$.
- In each of the remaining $r-3$ minutes, flip the first k switches that have never been flipped before.

This strategy turns on the remaining $k(r-3)$ lights, and Elizabeth escapes the room after r minutes.

b. First, if k divides n, then $E(n,k) = \frac{n}{k} = \lceil \frac{n}{k} \rceil$. Assume then that k does not divide n. Then let $r = \lceil \frac{n}{k} \rceil$, which implies $(r-1)k < n < rk$.

Because $(r-1)k < n$, it follows that $r-1$ minutes are not enough to flip each switch once, so $E(n,k) \geq r$.

Exactly one of $rk - n$ and $(r+1)k - n$ is even. There are two cases.

Case 1: Suppose $rk - n = 2b$ for some integer $b \geq 1$. As in the solution to Problem 6a, $b < k$. Then the following strategy turns all of the lights on in r minutes.

- In each of the first three minutes, flip switches 1 through b, along with the next $k - b$ switches that have not yet been flipped. This leaves $b + 3(k-b)$ lights on, and the rest off. Note that $b + 3(k-b) = 3k - 2b = 3k - (rk - n) = n - k(r-3)$.
- In each of the remaining $r - 3$ minutes, flip the first k switches that have never been flipped before.

This strategy turns on the remaining $k(r-3)$ lights, and Elizabeth escapes the room after r minutes.

Case 2: Suppose $(r+1)k - n = 2b$ for some integer $b \geq 1$. As in the solution to Problem 6a, $b < k$. Then the following strategy turns all of the lights on in $r+1$ minutes.

- In each of the first three minutes, flip switches 1 through b, along with the next $k - b$ switches that have not yet been flipped. This leaves $b + 3(k-b)$ lights on, and the rest off. Note that $b + 3(k-b) = 3k - 2b = 3k - (rk + k - n) = n - k(r-2)$.
- In each of the remaining $r - 2$ minutes, flip the first k switches that have never been flipped before.

This strategy turns on the remaining $k(r-2)$ lights, and Elizabeth escapes the room after $r+1$ minutes.

7. Consider the parity of n and of k. If n is odd and k is even, then as shown in Problem 3, $E(n,k) = \infty \neq 3$. If n is even and k is odd, then as noted in the solution to Problem 5, $E(n,k)$ is even, so $E(n,k) \neq 3$. If n and k are either both even or both odd and $\frac{n}{3} \leq k < n$, then the argument in Problem 4 shows that $E(n,k) = 3$.

If $k < \frac{n}{3}$, then three minutes is not enough time to flip each switch at least once, so $E(n,k) > 3$.

If $k = n$, then $E(n,k) = 1$.

Therefore $E(n,k) = 3$ if and only if $n + k$ is even (that is, n and k have the same parity) and $\frac{n}{3} \leq k < n$.

8. **a.** $I(6,3) = 0$. By definition, $I(6,3) = E(6,3) - \frac{6}{3}$. Because $3 \mid 6$, $E(6,3) = \frac{6}{3} = 2$, and so $I(6,3) = 2 - 2 = 0$.

b. $I(5,3) = \frac{4}{3}$. By definition, $I(5,3) = E(5,3) - \frac{5}{3}$. By Problem 2b, $E(5,3) = E(3 \cdot 1 + 2, 3) = 1 + 2 = 3$, and so $I(5,3) = 3 - \frac{5}{3} = \frac{4}{3}$.

c. One such pair is $(n,k) = (18,16)$. If $I(n,k) = \frac{15}{8}$, then $E(n,k) - \frac{n}{k} = \frac{15}{8}$. Note that $E(n,k) > 2$ because $k < n$.
Suppose $E(n,k) = 3$. Then $\frac{n}{k} = \frac{9}{8}$. From Problem 7, $E(n,k) = 3$ if and only if n and k have the same parity and $\frac{n}{3} \leq k < n$, so let $n = 18$ and $k = 16$. Then, $I(18,16) = E(18,16) - \frac{18}{16} = 3 - \frac{9}{8} = \frac{15}{8}$, as desired.

d. Let $n = 2x$ and $k = 2x - 1$ for some positive integer x. Then n is even and k is odd, so Problem 4 applies, and $E(n,k) = E(n, n-k) = E(n,1) = 2x$. Therefore $I(n,k) = 2x - \frac{2x}{2x-1}$. Because $x > 2$, it follows that $\frac{2x}{2x-1} < 2 < x$, so $I(n,k) > x$, as desired.

9. The least upper bound of S is 2. First, note that Problems 6a and 6b together show that if $k < \frac{n}{2}$, then $E(n,k)$ is either $\lceil \frac{n}{k} \rceil$ or $\lceil \frac{n}{k} \rceil + 1$ (or ∞ in the case where k is even and n is odd). Therefore $I(n,k) \leq 2$ because $E(n,k)$ is at most $\lceil \frac{n}{k} \rceil + 1$ and $\lceil \frac{n}{k} \rceil + 1 - \frac{n}{k} \leq 2$.

Now it will be shown that $I(n,k)$ can be arbitrarily close to 2. To do this, use values of n and k that are both odd, such that $\lceil \frac{n}{k} \rceil$ is even. Specifically, let $n = 3k + 2$ for k odd; then $\lceil \frac{n}{k} \rceil = 4$. Therefore $E(n,k) = 5$ by Problem 6b, and

$$I(n,k) = 5 - \frac{3k+2}{k} = 2 - \frac{2}{k}.$$

Letting k be a large odd integer, this gives an ordered pair (n,k) with $I(n,k)$ arbitrarily close to 2, as desired.

Combining these two claims shows that the least upper bound of S is 2.

10. It should be noted that integer answers are not possible because no positive integer can be the inefficiency of an ordered pair (n,k). This is because if $I(n,k) = E(n,k) - \frac{n}{k}$ is an integer, then $\frac{n}{k}$ is an integer so $E(n,k) = \frac{n}{k}$ and $I(n,k) = 0$.

To find non-integral rational numbers that cannot be the inefficiency of any ordered pair, start with a simple lemma.

Lemma: Given positive integers n, k, and r, $E(rn, rk) \le E(n,k)$.

Proof: If n is odd and k is even, then $E(n,k) = \infty$ and the lemma is trivially true. Otherwise there is a sequence of $E(n,k)$ sets of switches to flip which will result in escape. Partition the rn switches into r groups of n switches, and apply the strategy within each of the r groups, which uses rk flips per minute. This will achieve escape in $E(n,k)$ minutes, so $E(rn,rk)$ is at most $E(n,k)$. □

The values $E(n,k)$ and $E(rk,nk)$ can certainly equal one other, as when n is a multiple of k. The inequality can also be strict. For example, $E(4,3) = E(4,1) = 4$ but $E(8,6) = 3$ are obtained from values from earlier work.

One implication of the above lemma is that $I(n,k) \ge I(rn,rk)$.

Now all rational numbers q with $0 < q < 1$ are the inefficiency of some ordered pair (n,k). To show this, let $r = 3 - q = \frac{a}{b}$ in lowest terms. Then $r > 2$ so $a > 2b$, and by Problem 6a, $E(2a,2b) = \lceil \frac{a}{b} \rceil = 3$ and $I(2a,2b) = 3 - \frac{a}{b} = q$.

Next, all rational numbers q with $1 < q < 2$ are the inefficiency of some ordered pair (n,k). Again, let $3 - q = \frac{a}{b}$ in lowest terms. The fact that $1 < q < 2$ implies that $1 < \frac{a}{b} < 2$. So by Problem 4a, $E(2a,2b) = 3$ and $I(2a,2b) = 3 - \frac{a}{b} = q$.

If $\frac{n}{k} = 2$, then $I(n,k) = 0$ as noted above, while if $\frac{n}{k} > 2$, Problem 6b concludes that $E(n,k) = \lceil \frac{n}{k} \rceil$ or $E(n,k) = \lceil \frac{n}{k} \rceil + 1$. In either case, $I(n,k) \le \lceil \frac{n}{k} \rceil + 1 - \frac{n}{k} < 2$. As all nonnegative rational numbers less than 2 (except the excluded integer 1) have already been shown to be inefficiencies of ordered pairs, the cases $\frac{n}{k} \ge 2$ can now be excluded from consideration.

Consider ordered pairs (n,k) with $1 < \frac{n}{k} < 2$. If $n + k$ is even, Problem 4a implies that $E(n,k) = 3$ so $I(n,k) = 3 - \frac{n}{k} < 2$ and once again, all such numbers are already known to be inefficiencies. Of course if n is odd and k is even, then $I(n,k)$ is undefined.

Combining these results shows that a rational number will not be the inefficiency of some ordered pair simply whenever it is greater than 2 and avoids being the inefficiency of any ordered pair (n,k) where $1 < \frac{n}{k} < 2$, n is even, and k is odd.

Let $\frac{a}{b} > 2$ be a non-integral rational number in lowest terms. If $\frac{a}{b}$ is the inefficiency of some ordered pair (n,k), then $\frac{a}{b} = E(n,k) - \frac{n}{k}$. Because $E(n,k)$ is an integer, it follows that $\frac{a}{b} + \frac{n}{k}$ is also an integer. Because $1 < \frac{n}{k} < 2$, there is exactly one integer that it can be.

To find non-integral rational numbers that are not the inefficiency of any ordered pair (n,k), choose a rational number $\frac{a}{b}$ in such a way that the only possible choice for $\frac{n}{k}$, in lowest terms, leads to $I(n,k) < \frac{a}{b}$. Then because $I(rn,rk) \le I(n,k)$, no choice of n and k can possibly lead to $I(n,k) = \frac{a}{b}$.

One such choice is $\frac{a}{b} = \frac{11}{3}$. The only integer between $\frac{11}{3} + 1$ and $\frac{11}{3} + 2$ is $\frac{11}{3} + \frac{4}{3} = 5$. Choosing $n = 4$ and $k = 3$ satisfies the conditions that n is even, k is odd, and $1 < \frac{n}{k} < 2$, and also leads to $I(4,3) = E(4,3) - \frac{4}{3} = E(4,1) - \frac{4}{3} = 4 - \frac{4}{3} = \frac{8}{3} < \frac{11}{3}$. Thus no ordered pair (n,k) in the ratio $\frac{n}{k} = \frac{4}{3}$ can yield an inefficiency of $\frac{11}{3}$.

Another choice that will work is $\frac{a}{b} = \frac{17}{5}$. The only $\frac{n}{k}$ that could yield such an inefficiency would be $\frac{8}{5}$ (again, note that n is even, k is odd, and $1 < \frac{n}{k} < 2$). Then $I(8,5) = E(8,5) - \frac{8}{5} = 4 - \frac{8}{5} = \frac{12}{5} < \frac{17}{5}$, and this is a similar contradiction.

Note that there are many other non-integral rational numbers that are not the inefficiency of any ordered pair. They just aren't covered by the method outlined above and require different techniques. For instance, $\frac{5}{2}$ is such a number. For such an ordered pair (n, k) would require $\frac{n}{k}$ to be an odd multiple of $\frac{1}{2}$. Now n cannot be odd with k even; in that case, $I(n, k)$ is not defined. So both n and k are even. But then previous work shows that $I(n, k) < 2$, so $\frac{5}{2}$ is not an inefficiency of any ordered pair (n, k).