**Connecticut ARML Qualification Test, 2024**
**Solutions**

**1.** Find the last digit of $9^{8^7}$.

Solution:

The last digit of $9^n$ is 1 if $n$ is even, 9 if $n$ is odd. $8^7$ is even, therefore the answer is 1.

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**2.** Suppose the function $g(x) = f(x) + x^3$ is even and $f(-10) = 2024$. What is the value of $f(10)$ ?

Solution:

From the definition of even function, we have $g(10) = g(-10)$, that is:

$f(10) + 10^3 = f(-10) - 10^3 = 2024 - 1000 = 1024.$

Therefore $f(10) = 1024 - 1000 = 24.$

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**3.** An equilateral triangle and a regular hexagon have equal perimeters. If the area of the triangle is 20 square units, what is the area of the hexagon?

Solution:

Let the regular hexagon be given by $ABCDEF$. Extend $BA$ and $EF$ to intersect at $G$. Then $BEG$ is an equilateral triangle with the same perimeter as $ABCDEF$, with 2/3 the area. Therefore the answer is 30.

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**4.** A sequence $\{a_n\}$ is defined by $a_{n+2} = 2a_{n+1} + a_n$ (for $n \geq 1$), and $a_6 = 181$, $a_7 = 437$. Find the value of $a_4$.

Solution:

$a_7 = 2a_6 + a_5$, therefore $a_5 = 437 - 181 \cdot 2 = 75.$

$a_6 = 2a_5 + a_4$, therefore $a_4 = 181 - 75 \cdot 2 = 31.$

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**5.** Suppose that $a$ and $b$ are real numbers satisfying $\log_8 a^2 + \log_4 b^3 = 6$ and $\log_4 a^3 + \log_8 b^2 = 7$. Compute $ab$.

Solution:

We make use of $\log_{(y^n)} x = \log_y x^{\frac{1}{n}}$:

$\log_8 a^2 + \log_4 b^3 = \log_2 a^{\frac{2}{3}} + \log_2 b^{\frac{3}{2}} = \log_2 (a^{\frac{2}{3}} b^{\frac{3}{2}}) = 6$, so $a^{\frac{2}{3}} b^{\frac{3}{2}} = 2^6$.

$\log_4 a^3 + \log_8 b^2 = \log_2 a^{\frac{3}{2}} + \log_2 b^{\frac{2}{3}} = \log_2 (a^{\frac{3}{2}} b^{\frac{2}{3}}) = 7$, so $a^{\frac{3}{2}} b^{\frac{2}{3}} = 2^7$.

Multiplying the two equations together:

$(ab)^{\frac{13}{6}} = 2^{13}$, therefore $ab = 2^6 = 64$.

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**6.** Suppose that $pq + qr = 243$ for certain primes $p$, $q$, and $r$. Find $pqr$.

Solution:

$q(p + r) = 3^5$, we must have $q = 3$ and $p + r = 81$. Since two odd primes add to an even number, we must have $p = 2$ and $r = 79$ or vice versa. Therefore $pqr = 474$.

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**7.** Compute the sum of the roots of the polynomial

$(x + 1)^{2024} + (x - 2)^{2024} + (x + 3)^{2024} + (x - 4)^{2024} + \dots + (x - 2024)^{2024}$.

Solution:

Write this polynomial as $a_{2024}x^{2024} + a_{2023}x^{2023} + \dots + a_1 x + a_0$, then the sum of the roots is equal to $-\frac{a_{2023}}{a_{2024}}$.

We have $a_{2024} = 2024$,

$a_{2023} = 2024 \cdot (1 - 2 + 3 \dots + 2023 - 2024) = 2024 \cdot (-1012)$.

Therefore the answer is 1012.

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**8.** Suppose that the following function is defined for $0 < x < \frac{\pi}{2}$. What is the minimum value of the function?

$$f(x) = \frac{1+\tan^2 x}{(\sin^3 x)(\cos x)}$$

Solution:

The function simplifies to $f(x) = \frac{1}{(\sin^3 x)(\cos^3 x)}$. The denominator is $((\sin^2 x)(\cos^2 x))^{\frac{3}{2}}$.
With $\sin^2 x + \cos^2 x = 1$, the denominator is maximized when $\sin x = \cos x = \frac{\sqrt{2}}{2}$, giving $\min(f) = \frac{1}{(\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2})^3} = 8$.

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**9.** Find the remainder when $1! + 2! + 3! + ... + 100!$ is divided by 16.

Solution:
When $n >= 6$, $n!$ is divisible by 16. $1 + 2 + 6 + 24 + 120 = 153 \equiv 9 \ (mod\ 16)$.

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**10.** Let $\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\left(1 - \frac{1}{5^2}\right) ... \left(1 - \frac{1}{98^2}\right) = \frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Compute $m + n$.

Solution:

The $j$-th term in the product is $1 - \frac{1}{j^2} = \frac{j^2-1}{j^2} = \frac{j-1}{j} \cdot \frac{j+1}{j}$. The second factor in the $j$-th term therefore cancels the first factor in the $(j + 1)$-th term when multiplied. The whole expression therefore simplifies to $\frac{2}{3} \cdot \frac{99}{98} = \frac{33}{49}$. The answer is $33 + 49 = 82$.

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**11.** When the number $222...222_3$, with 1000 occurrences of the digit 2, is converted to base 9, the sum of the digits of the resulting base-9 number is $S$. Find $S$. (Your answer should be expressed in base 10.)

Solution:

$222...222_3 = 2(1+3^2+3^3+...+3^{999})$. Using the formula sum of a geometric series, this equals $2(3^{1000}-1)/(3-1) = 3^{1000}-1 = 9^{500}-1$, which expressed in base 9 is $888...888_9$ with 500 8s. The answer is 500 x 8 = 4000

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**12.** Alison, Justin, and Helen inherit their grandfather's flock of *n* emus. (Note: An emu is a type of bird.) According to the will, Alison is to receive $\frac{1}{2}$ of the emus, Justin is to receive $\frac{1}{3}$ of the emus, and Helen is to receive $\frac{1}{h}$ of the emus, where *h* is a positive integer. Unfortunately, *n* is not divisible by 2, 3, or *h*, and individual emus are not amenable to division, so the children are stuck until a neighbor gives them an emu. The increased flock is much easier to divide: Alison gets exactly $\frac{1}{2}$ of the emus, Justin gets exactly $\frac{1}{3}$ of the emus, and Helen gets exactly $\frac{1}{h}$ of the emus. Even better, there is exactly one emu left over, which they give back to the neighbor. Compute the greatest possible value of *n*.

<u>Solution:</u>

After the neighbor gives the group an emu, the neighbor gets one emu back, which is $\frac{1}{n+1}$ of the flock with an extra emu so $\frac{1}{n+1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{h} = 1$, or $\frac{1}{h} + \frac{1}{n+1} = \frac{1}{6}$. Clearly h>6, and the smaller than h is, the greater n is. Trying the smallest possible value of h=7, we get $\frac{1}{n+1} = = -\frac{1}{7} = \frac{1}{42}$, so n+1 = 42. The answer is 41.

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**13.** Circles of radius 5, 5, 8 and *r* are mutually externally tangent, where *r* = *m*/*n* for relatively prime positive integers *m* and *n*. Find *m* + *n*.

<u>Solution:</u>

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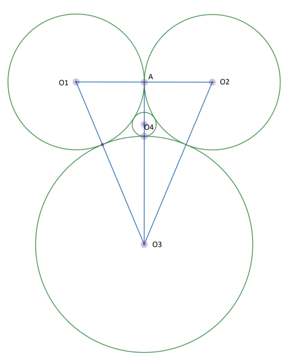
|---|---|
| <br> | Connect the radii of the circles. Note that $O_1AO_3$ form a 5-12-13 right triangle, where $AO_3 = AO_4 + O_4O_2 = 12$. $O_4O_2 = 8+r$ and from Pythagorean theorem, $AO_4 = \sqrt{(5+r)^2 - 5^2}$<br>We can thus solve for r from the equation<br>$\sqrt{(5+r)^2 - 5^2} + 8 + r = 12$, and get that $r = \frac{8}{9}$<br>The answer is 8+9 = 17 |

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**14.** Three standard, fair, six-sided dice are rolled. Given that the sum of the values rolled is 11, the probability that none of the numbers showing is prime can be expressed as $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Compute $m + n$.

<u>Solution:</u>

The number of ways 3 die rolls can sum to 11 is

$\sum_{i=1}^{6}$ (# ways to get 2 dice to sum to 4+i) =4 + 5 + 6 + 5 + 4 + 3 = 27

If none of the die rolled are prime, the only possible combination that sums to 11 is rolling a 1, 4, and 6. (You can try cases where you have two of the same nonprime number and see no other case works). There are 6 permutations this could occur. $\frac{6}{27} = \frac{2}{9}$ so the answer is 2+9 = 11.

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**15.** A bag contains eight cards numbered 1, 2, 3, …, 8. Alicia, Brian, Chris, and Damon each take two cards from the bag without looking (and without replacing the cards), and add the numbers on their cards. The probability that all four sums are odd is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

**Solution #1**

For all sums to be odd, each person will need to take one odd and one even card. For Alicia, after she takes the first card, there will be four cards of the opposite parity and three of the same parity remaining in the bag, so the probability that her second card is the opposite parity is 4/7. Similarly, given that Alicia succeeded in taking one of each parity card, Brian has a 3/5 probability of grabbing cards of opposite parity. Then Chris will succeed with a probability of 2/3 and Damon will always succeed. Thus the probability that all four succeed is the product of

$$\frac{4}{7} \times \frac{3}{5} \times \frac{2}{3} = \frac{8}{35}$$

Answer is 43.

**Solution #2**

Similarly as solution #1, for all sums to be odd, each person will need to take one odd and one even card. We can arrange the four odd cards in 4! ways, and similarly four even cards in 4! ways. Alicia, Brian, Chris, and Damon then can take each pair by sequence. However, between each pair, we can re-order the odd and even cards, so we need to multiple by 2 for each person.

Probability = $\frac{4!\times4!\times2^4}{8!} = \frac{8}{35}$

Answer is 43.

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**16.** $\left(\frac{\sqrt{6}+\sqrt{6}i}{\frac{3\sqrt{2}}{2}+\frac{\sqrt{6}}{2}i}\right)^4$ is $a+bi$ for some real $a$, $b$. Find $a^2 + b^2$.

Solution:

This expression is equal to $\left(\frac{2\sqrt{3}e^{i\pi/4}}{\sqrt{6}e^{i\pi/6}}\right)^4 = (\sqrt{2})^4 * e^{4i(\pi/4-\pi/6)} = 4e^{i\pi/3} = 2 + 2\sqrt{3}i$

The answer is $2^2 + (2\sqrt{3})^2 = 4 + 12 = 16$

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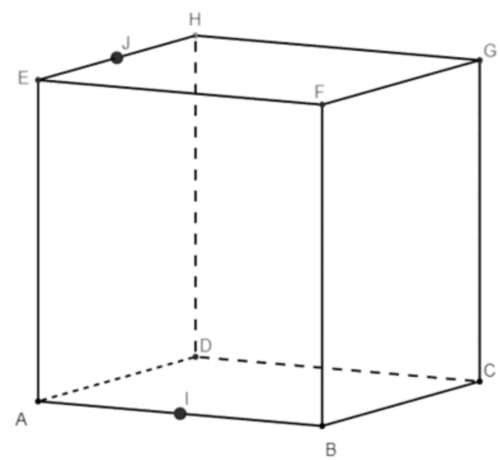
**17.** An ant is crawling on the faces of a solid cube with side length 20. (See the diagram below.) Its path starts from the midpoint $I$ of edge $\overline{AB}$ and ends at the midpoint $J$ of edge $\overline{EH}$. Let the shortest distance that the ant needs to travel be *d*. Find the integer closest to *d*.




Solution:

Between two points on a plane, the shortest distance is the line segment between them. There are three ways to flatten two faces of the cube so that $I$ and $J$ are on the same plane: (a) Flatten $AEFB$ and $AEHD$ along $AE$. (b) Flatten $EFBA$ and $EFGH$ along $EF$. (c) Flatten $ADHE$ and $ADCB$ along $AD$. In case (a), $IJ = \sqrt{20^2 + 20^2} = \sqrt{800}$. In case (b), $IJ = \sqrt{30^2 + 10^2} = \sqrt{1000}$. Case (c) gaves the same distance as case (b). Therefore (a) gives the shortest distance, with $d = 28$. This path goes through the midpoint of $AE$.

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**18.** Rectangle $ABCD$ has $AB = 8$ and $BC = 6$. Triangle $AEC$ is an isosceles right triangle with hypotenuse $\overline{AC}$ and $E$ on the same side of $\overline{AC}$ as point $B$. Triangle $BFD$ is an isosceles right triangle with hypotenuse $\overline{BD}$ and $F$ on the same side of $\overline{BD}$ as point $C$. Find the area of the quadrilateral $BCFE$.

<u>Solution:</u>

Place the center of $ABCD$ at the origin so that $B$ is at (4,3) and $C$ is at (4,-3). Then $E$ is at (3,4) and $F$ is at (3,-4). $BCFE$ is a trapezoid with bases 6 and 8 and height 1. Its area is therefore 7.

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**19.** Two ellipses, given by the equations $\frac{x^2}{2} + \frac{y^2}{4} = 1$ and $\frac{(x-1)^2}{9} + y^2 = 1$, intersect at four points. The center of the circle that passes through these four points is at $\left(\frac{1}{k}, 0\right)$. Find $k$.

<u>Solution:</u>

Write the two equations as: $2x^2 + y^2 = 4$ and $x^2 - 2x + 9y^2 = 8$. Multiply the first equation by 8 and add to the second equation to equalize the coefficients of $x^2$ and $y^2$ terms, we get: $17x^2 - 2x + 17y^2 = 40$, which is the equation of a circle. Any point that is an intersection of the two ellipses must satisfy both of the original equations, therefore will satisfy the equation of the circle. Rewrite the circle's equation as: $17(x - \frac{1}{17})^2 + 17y^2 = 40 - \frac{1}{17}$. We have $k = 17$.

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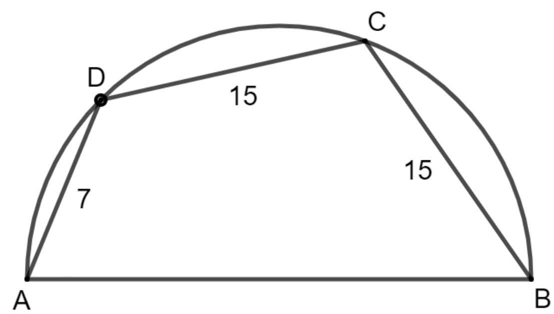
**20.** As shown in the diagram below, line segment *AB* is diameter of semicircle ω. Points *C* and *D* are on the semicircle with *AD* = 7, *CD* = *CB* = 15. Find the length of diameter *AB*.




<u>Solution:</u>
Let $O$ be the middle point of $AB$ and let $\theta = \angle AOD$, $\phi = \angle DOC = \angle COB$. Let $r = AO$ be the radius.

From the cosine formula we have: $\cos\theta = 1 - \frac{7^2}{2r^2}$, $\cos\phi = 1 - \frac{15^2}{2r^2}$. Using the fact that $\theta + 2\phi = \pi$, we have: $\cos\theta = -\cos 2\phi = -(2\cos^2\phi - 1)$. This gives us an equation involving $r$:

$$1 - \frac{7^2}{2r^2} = -1 + \frac{2\cdot 15^2}{r^2} - \frac{15^4}{2r^4}.$$

Simplifying, we get:

$$4r^4 - (7^2 + 4\cdot 15^2)r^2 + 15^4 = 0.$$

This is a quadratic equation of $r^2$, whose solutions are $r = 9$ and $r = \frac{25}{2}$. The first one leads to an obtuse $\phi$ and is therefore invalid. The second solution gives $AB = 25$.

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**21.** Five identical black marbles and seven identical white marbles are arranged in a line. How many distinguishable arrangements are there such that every black marble is next to a white marble?

**Solution #1**

Imagine the white marbles in fixed positions, forming 8 slots between them (including the two on either end) for the black marbles to go in. At most one black marble can go into the end slots and at most two into the middle slots.

There are three cases:

(a) All marbles go into different slots: $C(8, 5)$= 56 possibilities.

(b) One pair of marbles goes in one slot, and the rest go in different slots: 6 possibilities for which slot to put the pair in, and $C(7, 3)$= 35 possibilities for the last 3 marbles, total of $6 \times 35 = 210$.

(c) Two pairs of marbles each go into one slot, and the other marble in a different slot: $C(6, 2) = 15$ possibilities for the two pairs and 6 for the last marble, total of $6 \times 15 = 90$.

The total is 56 + 210 + 90 = 356

**Solution #2**

Count patterns that do not satisfy the requirement.

- *BBBBB*:  8 ways
- |BBBBW*, *WBBBB|:  14 ways
- *WBBBBW*:  42 ways
- |BBBW*, *WBBB|:  56 ways
- *WBBBW*:  168 ways
- |BBW*, *WBB|:  168 ways
- |BBW*BBB*, *BBB*WBB|:  -14 ways  (double-counting)
- |BBW*WBB|:  -6 ways (double-counting)

Sum of above = 436 ways

Total arrangements: 12!/(5!×7!) = 792 ways

Answer = 792 - 436 = 356

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**22.** The roots of the polynomial $x^3 + px^2 + 24x + q$ are integers, not necessarily distinct. Suppose that 2 is one of the roots. Compute the greatest possible value of $|q|$.

Solution:

Let a, b, c be the roots of the above polynomial, and WLOG let a=2. Then by Vieta's theorem ab+bc+ac = 24, so 2(b+c)+bc = 24, thus 4+2(b+c)+bc = 28. We can factor this as (b+2)(c+2) = 28. By Vieta's theorem we know $|q| = |abc|$, and since b and c are integers, we can try all combinations of integers where (b+2)(c+2) = 28 and find $b = -30$, $c = -3$ produces the greatest value of $|q|$, 180.

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**23.** George has 3 chickens on his farm. Each day, each chicken has a ⅓ chance of escaping. Each chicken's behavior is independent of the behavior of the other chickens and independent of its own behavior on previous days. Let $X$ be the whole number of days until all of George's chickens have escaped Then the expected value of $X$ is $\frac{p}{q}$, where $p$ and $q$ are relatively prime positive integers. Find $p + q$.

Solution:

We can create the following system of equations where $x_n$ is the expected number of days George's chickens all disappear when there are n chickens left.

$$x_3 = 1 + 1/27 * 0 + 2/9 * x_1 + 4/9 * x_2 + 8/27x_3$$
$$x_2 = 1 + 1/9 * 0 + 4/9 * x_1 + 4/9 * x_2$$
$$x_1 = 1 + 1/3 * 0 + 2/3 * x_1$$

Solving the system, we get $x_1 = 3$, $x_2 = \frac{21}{5}$, $x_3 = \frac{477}{95}$

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**24.** A frog starts at the point (1, 1). Every second, if the frog is at point $(x, y)$, it moves to $(x + 1, y)$ with probability $\frac{x}{x+y}$ and moves to $(x, y + 1)$ with probability $\frac{y}{x+y}$ . The frog stops moving when its $y$-coordinate is 10. Let $p$ be the probability that, when the frog stops, its $x$-coordinate is strictly less than 22. Then $p = \frac{m}{n}$, where $m, n$ are relatively prime positive integers. Find $m + n$.

Solution:

The probability the frog ends up at coordinate (k,10) when it stops moving is exactly

$\frac{9}{(9+k-1)(9+k)}$ This is because there are (9+k-2) choose k ways of reaching (k,10), and for every path that ends at (k,10) the probability of selecting it is $\frac{9!(k-1)!}{(9+k)!}$. The probability the x-coordinate is less than 22 is equal to

$$\sum_{k=1}^{21} \frac{9}{(9+k-1)(9+k)} = 9 \sum_{k=1}^{21} \left(\frac{1}{9+k-1} - \frac{1}{9+k}\right) = 9\left(\frac{1}{9} - \frac{1}{30}\right) = \frac{7}{10}$$

The answer is 7+10 = 17

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**25.** For each subset $A \subseteq \{1, 2, ..., 20\}$, define $s(A) = \sum_{a \in A} a$, with $s(\phi) = 0$. Find the number of subsets $A$ with $s(A) \equiv 0 \ (mod\ 5)$.

Solution:

Let $M$ be the answer we are looking for.

Consider the generating function $f(x) = (1 + x)(1 + x^2) \cdots (1 + x^{20})$. Let its full expansion be written as:

$$f(x) = \sum_{k=0}^{210} c_k x^k,$$

then $c_k$ counts the number of subsets $A$ with $s(A) = k$. Therefore we have:

$$M = \sum_{5 \mid k} c_k.$$

Next, we pick a complex number $z \neq 1$ such that $z^5 = 1$. We have two equality relationships:

$$1 + z + z^2 + z^3 + z^4 = 0, \qquad (1)$$

$$x^5 - 1 = (x - z)(x - z^2)(x - z^3)(x - z^4)(x - z^5). \qquad (2)$$

Note that if $5 \mid k$, then $1 = z^{k} = z^{2k} = z^{3k} = z^{4k}$ and they add up to 5. If $5 \nmid k$, then $\{1, z^{k}, z^{2k}, z^{3k}, z^{4k}\}$ is a permutation of $\{1, z, z^{2}, z^{3}, z^{4}\}$ and they add up to 0 because of Equation (1).
Therefore:

$$\begin{aligned} & f(1) + f(z) + f(z^{2}) + f(z^{3}) + f(z^{4}) \\ & \qquad = \sum_{k=0}^{210} c_{k} \cdot (1 + z^{k} + z^{2k} + z^{3k} + z^{4k}) \\ & \qquad = \sum_{5 \mid k} c_{k} \cdot 5 = 5M. \end{aligned}$$

We have $f(1) = 2^{20} = 1024^{2}$.
To compute $f(z)$, note that

$$\begin{aligned} f(z) &= (1 + z)(1 + z^{2}) \cdots (1 + z^{20}) \\ &= [(1 + z)(1 + z^{2})(1 + z^{3})(1 + z^{4})(1 + z^{5})]^{4} \\ &= [(- 1 - z)(- 1 - z^{2})(- 1 - z^{3})(- 1 - z^{4})(- 1 - z^{5})]^{4} \\ &= [(- 1)^{5} - 1]^{4} \qquad \text{(by Equation (2))} \\ &= 2^{4}. \end{aligned}$$

Similarly, for $k \in \{2, 3, 4\}$, since $\{1, z^{k}, z^{2k}, z^{3k}, z^{4k}\}$ is a permutation of $\{1, z, z^{2}, z^{3}, z^{4}\}$, we have $f(z^{k}) = 2^{4}$ as well.
Putting it all together,

$$M = \frac{f(1)+f(z)+f(z^{2})+f(z^{3})+f(z^{4})}{5} = \frac{1024^{2}+4\cdot 2^{4}}{5} = 209728.$$

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