

Name: _____

Grade: _____

School: _____

CONNECTICUT ARML QUALIFICATION TEST

Thursday, March 7, 2024

Pre-test instructions:

- When you receive the link to this test, print out the test and staple it. Do not read it.
- Then gather all you will need for the test: pencils, scrap paper. (Graph paper and protractors are not allowed.)
- Put your phone, all calculators, and any other electronic devices apart from your computer on a surface way out of reach. (You will need your phone at the end of the test in order to take images of your scratchwork.)
- Go to the bathroom now, so that you won't need to during the test.
- Your computer should still be signed on to the Zoom meeting you have been given for this test. Position your computer camera on your desk about 20 inches to the side of the place you will be working. Your entire work area should be visible to the proctors, and it must **not** be possible for other participants to read what you are writing.
- Read the directions below and wait to be told to start the test.

Test directions:

- You will be given 1 hour 45 minutes in which to answer 25 questions.
- All questions carry equal weight.
- All answers are positive integers.
- Write your work in the space provided. You will be required to submit your work at the end of the test in order to confirm the authenticity of your answers. This scratchwork will not be graded.
- All the usual rules for testing apply to this test, including the fact that no communication of any sort with any person is allowed, except with a proctor of this test. (You will be asked to sign a pledge at the end of the test.)
- Books, class notes, etc. are **not** allowed.
- Calculators and the Internet are **not** allowed.

1. Find the last digit of 9^{8^7} .

2. Suppose the function $g(x) = f(x) + x^3$ is even and $f(-10) = 2024$. What is the value of $f(10)$?

3. An equilateral triangle and a regular hexagon have equal perimeters. If the area of the triangle is 20 square units, what is the area of the hexagon?

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4. A sequence $\{a_n\}$ is defined by $a_{n+2} = 2a_{n+1} + a_n$ (for $n \geq 1$), and $a_6 = 181$, $a_7 = 437$. Find the value of a_4 .
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5. Suppose that a and b are real numbers satisfying $\log_8 a^2 + \log_4 b^3 = 6$ and $\log_4 a^3 + \log_8 b^2 = 7$. Compute ab .

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6. Suppose that $pq + qr = 243$ for certain primes p , q , and r . Find pqr .
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7. Compute the sum of the roots of the polynomial

$$(x + 1)^{2024} + (x - 2)^{2024} + (x + 3)^{2024} + (x - 4)^{2024} + \dots + (x - 2024)^{2024}.$$

8. Suppose that the following function is defined for $0 < x < \frac{\pi}{2}$. What is the minimum value of the function?

$$f(x) = \frac{1 + \tan^2 x}{(\sin^3 x)(\cos x)}$$

9. Find the remainder when $1! + 2! + 3! + \dots + 100!$ is divided by 16.

10. Let $\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\left(1 - \frac{1}{5^2}\right) \dots \left(1 - \frac{1}{98^2}\right) = \frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

11. When the number $222\dots222_3$, with 1000 occurrences of the digit 2, is converted to base 9, the sum of the digits of the resulting base-9 number is S . Find S . (Your answer should be expressed in base 10.)

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12. Alison, Justin, and Helen inherit their grandfather's flock of n emus. (Note: An emu is a type of bird.) According to the will, Alison is to receive $\frac{1}{2}$ of the emus, Justin is to receive $\frac{1}{3}$ of the emus, and Helen is to receive $\frac{1}{h}$ of the emus, where h is a positive integer. Unfortunately, n is not divisible by 2, 3, or h , and individual emus are not amenable to division, so the children are stuck until a neighbor gives them an emu. The increased flock is much easier to divide: Alison gets exactly $\frac{1}{2}$ of the emus, Justin gets exactly $\frac{1}{3}$ of the emus, and Helen gets exactly $\frac{1}{h}$ of the emus. Even better, there is exactly one emu left over, which they give back to the neighbor. Compute the greatest possible value of n .
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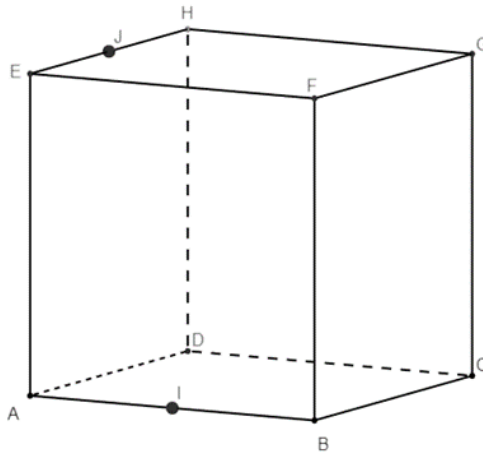
13. Circles of radius 5, 5, 8 and r are mutually externally tangent, where $r = m/n$ for relatively prime positive integers m and n . Find $m + n$.

14. Three standard, fair, six-sided dice are rolled. Given that the sum of the values rolled is 11, the probability that none of the numbers showing is prime can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

15. A bag contains eight cards numbered 1, 2, 3, ..., 8. Alicia, Brian, Chris, and Damon each take two cards from the bag without looking (and without replacing the cards), and add the numbers on their cards. The probability that all four sums are odd is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

16. $\left(\frac{\sqrt{6} + \sqrt{6}i}{\frac{3\sqrt{2}}{2} + \frac{\sqrt{6}}{2}i}\right)^4$ is $a+bi$ for some real a, b . Find $a^2 + b^2$.

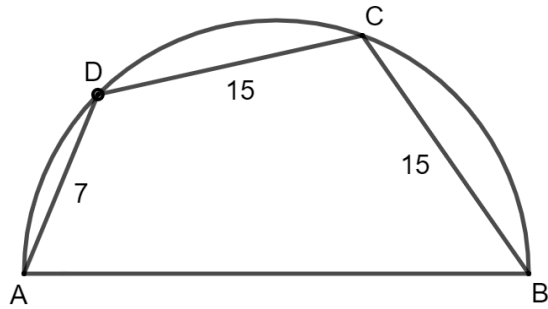
17. An ant is crawling on the faces of a solid cube with side length 20. (See the diagram below.) Its path starts from the midpoint I of edge \overline{AB} and ends at the midpoint J of edge \overline{EH} . Let the shortest distance that the ant needs to travel be d . Find the integer closest to d .



- 18.** Rectangle $ABCD$ has $AB = 8$ and $BC = 6$. Triangle AEC is an isosceles right triangle with hypotenuse \overline{AC} and E on the same side of \overline{AC} as point B . Triangle BFD is an isosceles right triangle with hypotenuse \overline{BD} and F on the same side of \overline{BD} as point C . Find the area of the quadrilateral $BCFE$.
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19. Two ellipses, given by the equations $\frac{x^2}{2} + \frac{y^2}{4} = 1$ and $\frac{(x-1)^2}{9} + y^2 = 1$, intersect at four points. The center of the circle that passes through these four points is at $(\frac{1}{k}, 0)$. Find k .
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20. As shown in the diagram below, line segment AB is diameter of semicircle ω . Points C and D are on the semicircle with $AD = 7$, $CD = CB = 15$. Find the length of diameter AB .



21. Five identical black marbles and seven identical white marbles are arranged in a line. How many distinguishable arrangements are there such that every black marble is next to a white marble?

22. The roots of the polynomial $x^3 + px^2 + 24x + q$ are integers, not necessarily distinct. Suppose that 2 is one of the roots. Compute the greatest possible value of $|q|$.
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23. George has 3 chickens on his farm. Each day, each chicken has a $\frac{1}{3}$ chance of escaping. Each chicken's behavior is independent of the behavior of the other chickens and independent of its own behavior on previous days. Let X be the whole number of days until all of George's chickens have escaped. Then the expected value of X is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.
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24. A frog starts at the point $(1, 1)$. Every second, if the frog is at point (x, y) , it moves to $(x + 1, y)$ with probability $\frac{x}{x+y}$ and moves to $(x, y + 1)$ with probability $\frac{y}{x+y}$. The frog stops moving when its y -coordinate is 10. Let p be the probability that, when the frog stops, its x -coordinate is strictly less than 22. Then $p = \frac{m}{n}$, where m, n are relatively prime positive integers. Find $m + n$.
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25. For each subset $A \subseteq \{1, 2, \dots, 20\}$, define $s(A) = \sum_{a \in A} a$, with $s(\emptyset) = 0$. Find the number of subsets A with $s(A) \equiv 0 \pmod{5}$.

END OF TEST

Post-test directions. You have 10 minutes to complete steps 1–3 below.

1. Enter your answers on the Google Form provided in the Zoom chat. Check that you have typed your answers correctly, and then submit the form. (You have 3½ minutes to do this.)
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2. Please complete the following pledge by crossing out the words that do not apply, and add your signature.

I did / did not abide by exam rules exactly as is expected in a classroom.

I did / did not complete this test without the help of any person or source.

I did / did not complete this test without the use of a calculator and/or the Internet.

Signed: _____

3. Get your phone, and using the Adobe Scan app, create a PDF consisting of all the pages of this test, including the front page (which includes your name) and this page (which includes your signed pledge). For this document, select share, email, and send the link by email to CTARMLTeam@gmail.com. Please include your name in the subject line.
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4. Please remain on the Zoom meeting until you are told that you may leave.
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