CONNECTICUT ARML QUALIFICATION TEST, 2025
Solutions

1. There are 10 delegates at a conference: 6 from Party A and 4 from Party B. A committee of 6 delegates is to be formed such that, on the committee, the number of delegates from Party A is greater than the number of delegates from Party B. In how many ways can this be done?
[Answer: 115]

Solution:
Number of ways to choose 4 from Party A and 2 from Party B: $\binom{6}{4}\binom{4}{2} = 15 \cdot 6 = 90$.
Number of ways to choose 5 from Party A and 1 from Party B: $\binom{6}{5}\binom{4}{1} = 6 \cdot 4 = 24$.
Number of ways to choose 6 from Party A and 0 from Party B: 1.
Total: $90 + 24 + 1 = 115$.

---

2. Find the largest positive integer n such that $30!$ is divisible by 2^n.
[Answer: 26]

Solution:

$$\begin{aligned} n &= \left\lfloor \frac{30}{2} \right\rfloor + \left\lfloor \frac{30}{2^2} \right\rfloor + \left\lfloor \frac{30}{2^3} \right\rfloor + \left\lfloor \frac{30}{2^4} \right\rfloor + 0 \\ &= 15 + 7 + 3 + 1 \\ &= 26 \end{aligned}$$

---

3. Let α, β, and γ be the solutions of the equation $2x^3 + 3x^2 - 5x - 4 = 0$. Then $\alpha^2 + \beta^2 + \gamma^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
[Answer: 33]

Solution:
$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma) = \left(-\frac{3}{2}\right)^2 + 2 \cdot \left(\frac{5}{2}\right) = \frac{29}{4}$.
$29 + 4 = 33$.

---

4. Four circles of radius 4 are internally tangent to a circle of radius r, with each of the smaller circles externally tangent to two others. Then $r = a + b\sqrt{c}$, where a, b, and c are positive integers and c is not divisible by the square of any prime number. Find $a + b + c$.
[Answer: 10]

Solution. The centers of the 4 smaller circles form a rhombus, and for there to exist a larger circle that is externally tangent to all four, this rhombus has to be a square, by symmetry. Then, referring to the figure 1, $OC_2 = 4\sqrt{2}$ and

$$\begin{aligned} r &= 4 + OC_2 \\ &= 4 + 4\sqrt{2} \\ \text{Ans} &= 4 + 4 + 2 \\ &= 10 \end{aligned}$$

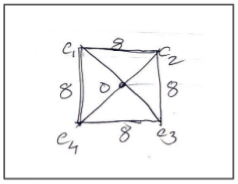


Figure 1: Figure for Q4

---

5. Find the number of values of x with $0 < x < 2\pi$ such that

$$(4^{\cos 2x})^{\sin 2x} = 2$$

[Answer: 4]

Solution:
$(4^{\cos 2x})^{\sin 2x} = 2 \Rightarrow 4^{\sin 2x \cos 2x} = 2 \Rightarrow \sin 2x \cos 2x = \frac{1}{2} \Rightarrow 2 \sin 2x \cos 2x = 1$
$\Rightarrow \sin 4x = 1 \Rightarrow 4x = \frac{\pi}{2} + 2k\pi \Rightarrow x = \frac{\pi}{8} + \frac{k\pi}{2} \Rightarrow 0 \le \frac{1}{8} + \frac{k}{2} \le 2 \Rightarrow 0 \le k \le 3.$
There are 4 values of x.

---

6. Let A, B, C, and D be positive integers (not necessarily distinct) such that $A^2 + B^2 = 20$ and $C^2 - D^2 = 24$. Find the greatest possible value for the sum $A + B + C + D$.
[Answer: 18]

Solution. Optimize $A + B$ and $C + D$ separately.

$$\begin{aligned} (A+B)^2 &= 2(A+B)^2 - (A-B)^2 \\ &= 40 - (A-B)^2 \end{aligned}$$

Since both $(A+B)^2$ and $(A-B)^2$ are perfect squares and $(A+B)$ and $(A-B)$ have the same parity, the minimum possible value of $(A-B)^2$ is 4, and the corresponding max value of $A+B$ is 6. Similarly, $(C+D)(C-D) = 24$ where $C+D$ and $C-D$ are positive integers of the same parity. So the max possible value of $C+D$ is 12.

Therefore the answer is $6 + 12 = 18$

---

7. Let f be a function with the property that, for any real x, $xf(x) + f(x+2) = x^2$. Find $f(8)$.
[Answer: 36]

Solution:
$x = 0 \Rightarrow f(2) = 0,$
$x = 2 \Rightarrow f(4) = 4,$
$x = 4 \Rightarrow f(6) = 0,$
$x = 6 \Rightarrow f(8) = 36$

---

8. In the complex plane, let z and w be numbers satisfying $z^6 = 1$ and $w^4 = -1$. Given that 0, z, w, and $z + w$ form a quadrilateral with nonzero area, the minimum possible area of the quadrilateral can be expressed as $\frac{\sqrt{a}-\sqrt{b}}{c}$, where a, b, and c are positive integers, and neither a nor b is divisible by the square of any prime number. Find $a + b + c$.
[Answer: 12]

Solution. z and w satisfy, $w = \pm\frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$ and $z = +1, -1,$ or, $\pm\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, which also satisfy $|z| = |w| = 1$. If θ is the angle between z and w, the area of the quadrilateral is $\sin\theta$.

The angles that z and w form with the positive x-axis are respectively $0°$, $\pm 60°$, $\pm 120°$ and $180°$ (for z) and $\pm 45°$ and $\pm 135°$ (for w). The minimum angle between these two sequences is $15°$ which also corresponds to the minimum area of $\sin 15°$.

To compute the actual value, we write $s = \sin 15°$, and note that

$$\begin{aligned} \frac{\sqrt{3}}{2} &= \cos 30° \\ &= 1 - 2s^2, \text{ Using the doubling formula for cosine and noting the definition of } s \end{aligned}$$

Solving this equation, we get $s = \frac{\sqrt{2-\sqrt{3}}}{2}$. To simplify this expression, we set $a = \sqrt{2-\sqrt{3}}$ and $b = \sqrt{2+\sqrt{3}}$, which satisfy $ab = 1$ and $a^2 + b^2 = 4$. Leading to $b + a = \sqrt{6}$ and $b - a = \sqrt{2}$, and finally to $a = \frac{\sqrt{6}-\sqrt{2}}{2}$. Substituting in the expression for s, we get $s = \frac{\sqrt{6}-\sqrt{2}}{4}$.

Finally, the answer is $6 + 2 + 4 = 12$

---

9. Point D lies inside triangle ABC so that $AD = 8$, $BD = 5$, and $\text{m}\angle ADC = \text{m}\angle ADB = \text{m}\angle CDB = 120°$.
Given that $\angle ABC$ is a right angle, find the length CD.
[Answer: 30]

<u>Solution:</u>
<u>Method 1</u>: Use cosine rule.
$AB^2 = 5^2 + 8^2 + 5 \cdot 8, \quad BC^2 = 5^2 + x^2 + 5x, \quad AC^2 = 8^2 + x^2 + 8x.$
$AB^2 + BC^2 = AC^2 \Rightarrow 3x = 90 \Rightarrow x = 30.$

<u>Method 2</u>: Set $D = (0,0), B = (0,5), A = (-4\sqrt{3}, -4), C = \left(\frac{\sqrt{3}}{2}x, -\frac{1}{2}x\right)$.
Line AB has slope $\frac{3\sqrt{3}}{4}$. Then line BC has slope $-\frac{4\sqrt{3}}{9}$. Calculating this slope using coordinates for B, C gives:

$$\frac{-\frac{1}{2}x - 5}{\frac{\sqrt{3}}{2}x} = -\frac{4\sqrt{3}}{9} \Rightarrow x = 30.$$

---

10. Let n be a positive integer. When n dice are rolled, the nonzero probability of obtaining a sum of 2025 is the same as the probability of obtaining a sum of S. As the number n varies, what is the smallest possible value of S?
[Answer: 341]

Solution. Since $P(2025) > 0$, we have $338 \le n \le 2025$. The event of getting a sum of k in n throws is bijectively mapped to the event of getting a sum of $7n - k$, by mapping $1 \leftrightarrow 6, 2 \leftrightarrow 5, 3 \leftrightarrow 4$.

So, $P_n(k) \le P_n(7n - k)$, and since the mapping is self-inverse, we also get $P_n(7n - k) \le P_n(k)$, and hence $P_n(7n - k) = P_n(k)$.

Therefore $\min S = 7n - 2025$ for the minimum value of n. Thus the result is $7 \times 338 - 2025 = 341$.

---

11. What is the only real number $x > 1$ that satisfies the equation below?

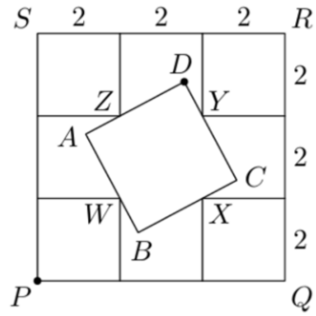
$$(\log_3 x)(\log_5 x)(\log_7 x) = (\log_3 x)(\log_5 x) + (\log_5 x)(\log_7 x) + (\log_3 x)(\log_7 x)$$

[Answer: 105]

<u>Solution:</u>
$\log_3 x \log_5 x \log_7 x = \log_3 x \log_5 x + \log_5 x \log_7 x + \log_3 x \log_7 x,$
Divide by $\log_3 x$, get:
$\log_5 x \log_7 x = \log_5 x + \log_5 3 \log_7 x + \log_7 x,$
Divide by $\log_5 x$, get:
$\log_7 x = 1 + \log_7 5 + \log_5 3 \log_7 5,$
$\Rightarrow x = 7^{1+\log_7 5 + \log_5 3 \log_7 5} = 7 \cdot 5 \cdot 3 = 105.$

---

12. In the corners of a square *PQRS* with side length 6, four smaller squares are placed with side lengths 2, as shown in the diagram below. We will label the points W, X, Y and Z as shown in the diagram. A square *ABCD* is constructed in such a way that the points W, X, Y, and Z lie on sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively. Find the maximum possible value of $(PD)^2$.



[Answer: 36]

Solution. The locus of D is the semi-circle with ZY as the diameter. The max PD^2 corresponds to the max PD, which occurs when D is the point on the semi-circle collinear with P and the center of the semi-circle, where the center of the circle is the midpoint of the segment ZY. Hence,

$$\begin{aligned} \max PD &= \sqrt{3^2+4^2} + \text{radius} \\ &= 5+1 \\ &= 6 \end{aligned}$$

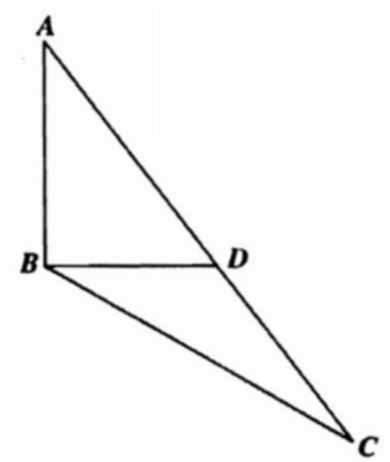
Hence, the answer is $6^2 = 36$

---

13. Given that x and y are real numbers and $x^2 + y^2 = 14x + 6y + 6$, what is the largest possible value of $3x + 4y$?
[Answer: 73]

Solution:
Method 1: Use analytical geometry.
The equation can be written as: $(x - 7)^2 + (y - 3)^2 = 64$, which is a circle of radius 8 centered at $C = (7,3)$.
The objective function to be maximized, $3x + 4y = a$, can be written as $y = -\frac{3}{4}x + \frac{a}{4}$, which is a straight line. The value of a is maximized when this line is tangent to the circle at a point P. Note that the slope of CP is $\frac{4}{3}$, which gives the coordinates of $P = \left(7 + \frac{24}{5}, 3 + \frac{32}{5}\right)$ with the radius being 8. Therefore the y-intercept of the line is given by:
$$\frac{a}{4} - \left(3 + \frac{32}{5}\right) = \frac{3}{4}\left(7 + \frac{24}{5}\right) \Rightarrow a = 73.$$
Method 2: Use quadratic equation discriminant.
From $3x + 4y = a$, we get $y = \frac{a}{4} - \frac{3}{4}x$. Substituting into $x^2 + y^2 = 14x + 6y + 6$ and simplifying, we get:
$$25x^2 - (6a + 152)x + (a^2 - 24a - 96) = 0.$$
At the maximum, this equation's discriminant is 0, which gives an equation for a, after simplification: $a^2 - 66a - 511 = 0$, or $(a + 7)(a - 73) = 0, \Rightarrow a = 73$.

14. In the diagram below, $AB = 4$, $BD = 3$, $AD = 5$, $m\angle DBC = 30°$, and $\overline{ADC}$ is a straight line. $DC = \frac{a\sqrt{b}+c}{d}$, where a, b, c, and d are positive integers, b is not divisible by the square of any prime number, and $\gcd(a, c, d) = 1$. Find $a + b + c + d$.



Solution. Let $CD = x$. The slope of the line CD is (ignoring sign) $\frac{4}{3}$, and corresponding cosine and sine are $\frac{3}{5}$ and $\frac{4}{5}$. Writing the slope of BC ($\tan 30° = \sqrt{3}$), we get

$$3 + \frac{3}{5}x = \sqrt{3}\frac{4}{5}x$$

Solving, we get $x = \frac{15}{4\sqrt{3}-3} = \frac{15(4\sqrt{3}+3)}{48-9} = \frac{5(4\sqrt{3}+3)}{13} = \frac{20\sqrt{3}+15)}{13}$.

Or, the answer is $20 + 15 + 13 + 3 = 51$

---

15. How many nonempty subsets of {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} have the property that no two elements of the subset differ by more than 5 ? For example, count the subsets {3}, {2, 5, 7} and {5, 6, 7, 8, 9}, but not the subset {1, 3, 5, 7}.
[Answer: 255]

Solution:
If the subset has 1 element, there are 12 possibilities.
For subsets S that have at least 2 elements, they can be grouped into disjoint cases based on $D = \max(S) - \min(S)$, with D ranging from 1 to 5. For each D, there can be $12 - D$ possibilities to chose $\min(S)$ and $\max(S)$ together, and 2^{D-1} possibilities for each element strictly between $\min(S)$ and $\max(S)$ to be included or not.
$12 + 7 \cdot 2^4 + 8 \cdot 2^3 + 9 \cdot 2^2 + 10 \cdot 2^1 + 11 \cdot 2^0 = 12 + 112 + 64 + 36 + 20 + 11 = 255$.

---

16. Suppose that $2\tan^{-1}(x) + \tan^{-1}(2x) = \frac{\pi}{2}$. Then $x^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
[Answer: 6]

Solution. Taking tangent of both sides, $\tan(2\tan^{-1} x) = \frac{1}{2x}$, or,

$$\frac{2x}{1-x^2} = \frac{1}{2x}$$

Or, $x^2 = \frac{1}{5}$
Answer $= \underline{6}$

---

17. The base-2 representation of the number N is

$$1\,0\,1\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,\underline{A}\,\underline{B}\,\underline{C}\,1\,1\,0$$

where each of the digits A, B, C is a 0 or a 1. If N is divisible by 7, what are the last seven digits of the base-2 representation of N? (The answer is a seven-digit number consisting of 1s and 0s, only.)
[Answer: 1010110]

Solution:
Since $8 \equiv 1 \bmod 7$, if $N = 8M + L$ then $N \bmod 7 \equiv (M + L) \bmod 7$.
Applying this relationship recursively, N mod 7 can be calculated by adding up each group of 3 digits of the number together, mod 7. Since ABC is the second such 3-digit group, we can calculate the sum of the other groups first:
$110_2 + 101_2 + 010_2 + 101_2 + 110_2 + 010_2 = 11010_2 \equiv (010 + 011) = 101 \bmod 7$.
Therefore $ABC = 010_2$ is the only choice to make $N \equiv 0 \bmod 7$.
The last 7 digits of N is 1010110.

---

18. In triangle ABC, $AB = AC$ and we let ω be the unique circle inscribed in the triangle. Suppose that the orthocenter of triangle ABC lies on ω. Then $\cos \angle BAC = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
[Answer: 10]

Solution. From the condition,
$\tan \angle HBD = 2 \tan \angle IBD$.
Since quadrilateral $B'ABD$ is cyclic, $\angle HBD = \frac{1}{2}\angle BAC$.
Also, $\angle IBD = \frac{1}{2}\angle ABC = 45^\circ - \frac{1}{4}\angle BAC$.
Using the condition on tangents, $\tan \frac{A}{2} = 2\tan(45^\circ - \frac{A}{4})$.
Let $t = \tan \frac{A}{4}$, then $\frac{2t}{1-t^2} = \frac{2(1-t)}{1+t}$.
Or, $t^2 - 3t + 1 = 0$, where we know that $0 < t < 1$. Solving and using this inequality, we get $t = \frac{3-\sqrt{5}}{2}$.

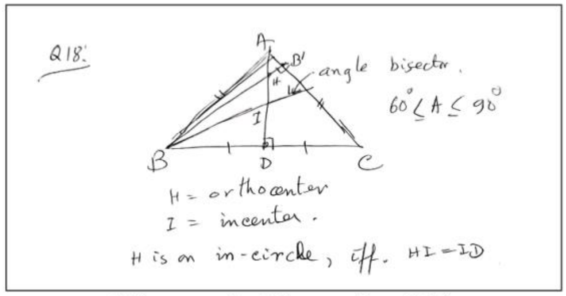


Figure 4: Figure for Q18

---

19. In the sequence $a_1, a_2, a_3, \ldots$, let $a_k = (k^2+1)k!$ and $b_k = a_1 + a_2 + \cdots + a_k$. Then

$$\frac{a_{100}}{b_{100}} = \frac{m}{n}$$

where m and n are relatively prime positive integers. Find $n - m$.
[Answer: 99]

<u>Solution:</u>
We calculate the first few terms of a_k for heuristics:
$a_1 = 2 \cdot 1!$,
$a_2 = 5 \cdot 2!$,
$a_3 = 10 \cdot 3!$,
$a_4 = 17 \cdot 4!$.
Note that:
$b_2 = a_1 + a_2 = 6 \cdot 2! = 2 \cdot 3!$,
$b_3 = b_2 + a_3 = 2 \cdot 3! + 10 \cdot 3! = 12 \cdot 3! = 3 \cdot 4!$,
$b_4 = b_3 + a_4 = 3 \cdot 4! + 17 \cdot 4! = 20 \cdot 4! = 4 \cdot 5!$.
We can therefore stipulate that $b_k = k(k+1)!$ and prove it by induction:
$b_k + a_{k+1} = k(k+1)! + ((k+1)^2+1)(k+1)! = [(k+1)^2 + (k+1)](k+1)!$
$= (k+1)(k+2)! = b_{k+1}$.
Therefore:
$\frac{a_{100}}{b_{100}} = \frac{10001 \cdot 100!}{100 \cdot 101!} = \frac{10001}{10100}$,
$10100 - 10001 = 99$.

---

20. Five people take turns rolling a fair six-sided die numbered from 1 to 6, with each person rolling exactly once. The probability that each person's roll is greater than or equal to the previous person's roll is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.
[Answer: 223]

Solution. $P = \frac{\text{number of non-decreasing roll sequences}}{6^5}$

Calculate numerator via stars and bars. Each roll is a bar. Each star represents an increment of 1 from the previous value, assuming we start at 1. Thus $2,2,4,5,6$ is represented by $+ - - + + - + - + -$, $1,2,3,3,5$ is represented by $- + - + - - + + - +$.

There are $\binom{10}{5}$ possibilities for the numerator. Hence

$$P = \frac{\binom{10}{5}}{216} = \frac{7}{216}$$

.

Answer: $7 + 216 = 223$.

You can also obtain the numerator using dynamic programming on a square grid 5 columns and 6 rows and starting from the top right.

---

21. We will call an n-digit number *sweet* if its n digits are an arrangement of the set $\{1, 2, \ldots, n\}$ and, for $k = 1, 2, \ldots n$, its first k digits form an integer that is divisible by k. For example, 321 is a sweet 3-digit integer because 1 divides 3, 2 divides 32 and 3 divides 321. How many sweet 6-digit numbers are there?
[Answer: 2]

Solution:
Let the number be $abcdef$.
$2 \mid ab \Rightarrow b \in \{2,4,6\}$,
$3 \mid abc \Rightarrow 3 \mid (a+b+c)$,
$4 \mid abcd \Rightarrow 4 \mid cd \Rightarrow d \in \{2,4,6\}$,
$5 \mid abcde \Rightarrow e = 5$,
$6 \mid abcdef \Rightarrow f \in \{2,4,6\}$.
Since $\{b,d,f\} = \{2,4,6\}$, we must have $\{a,c\} = \{1,3\}$. Then $3 \mid abc \Rightarrow b = 2$.
Then $4 \mid cd \Longrightarrow d = 6$.
Therefore, 123654 and 321654 are the only 2 possibilities.

---

22. A farmer has 5 cows, 4 pigs, and 7 horses. The farmer will sort the animals into pairs in such a way that no animal is paired with an animal of the same species. Assuming that all the animals are distinguishable, in how many ways can this be done?
[Answer: 100800]

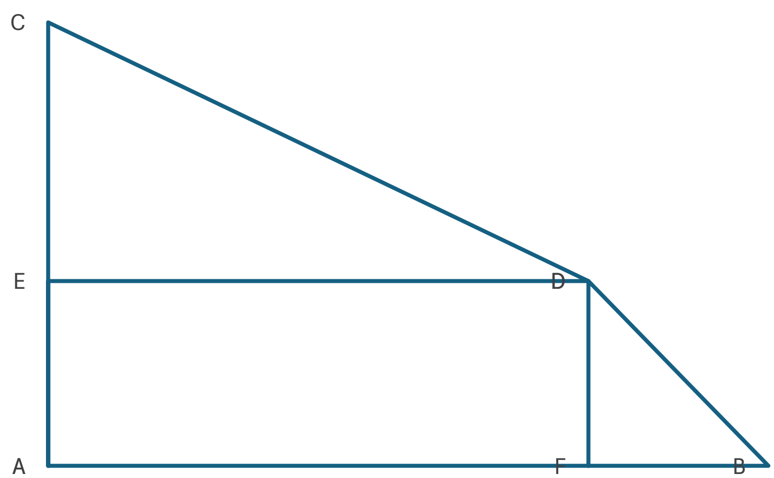
Solution. There are 16 animals, hence 8 pairs. The horses pair up with either a pig or a cow. Remaining 1 cow and 1 pig pair up. There are $5 \times 4 = 20$ ways to do this last pairing. Once this is set, the rest are equivalent to a permutation of the horses (since each animal is distinguishable) hence contribute $7!$ ways.

Answer $= 20 \times 7! = 100800$

---

23. The value of x that minimizes $\sqrt{x^2+49}+\sqrt{(8-x)^2+25}$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.
[Answer: 17]

Solution:



In the above, let: $AB = 8,\ AE = FD = 5,\ CE = 7,\ AF = ED = x$. Then:

$\sqrt{x^2+49} = CD,\qquad \sqrt{(8-x)^2+25} = BD.$

Their sum is minimized when BDC is a straight line. We have then:

$$\frac{ED}{CE} = \frac{AB}{AC} \Rightarrow \frac{x}{7} = \frac{8}{5+7} \Rightarrow x = \frac{14}{3}.$$

$14 + 3 = 17$.

---

24. The longer of the two side lengths of rectangle $AXCY$ is 11, and rectangle $AXCY$ shares diagonal $\overline{AC}$ with square $ABCD$. Assume that B and X lie on the same side of $\overline{AC}$ such that triangle BXC and square $ABCD$ are non-overlapping. The maximum area of triangle BXC across all such configurations is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.
[Answer: 137]

Solution. Let r be the radius of the circle of which $\overline{AC}$ is the diagonal.

Then, $2r = \sqrt{11^2 + t^2}$.

Therefore $a = \sqrt{2}r = \frac{\sqrt{11^2+t^2}}{\sqrt{2}}$

$\text{Area}(BXC) = A = \frac{ah}{2} = \frac{h\sqrt{11^2+t^2}}{2\sqrt{2}}$

Or, $A = \frac{t \sin\theta \sqrt{11^2+t^2}}{2\sqrt{2}}$

But, $\theta = 45^\circ - \theta$

So $\sin\theta = \frac{1}{\sqrt{2}}(\cos\alpha - \sin\alpha)$

Also, $\cos\alpha = \frac{11}{\sqrt{11^2+t^2}}$

$\cos\alpha = \frac{t}{\sqrt{11^2+t^2}}$

Substituting all in the formula for area

$A = \frac{t(11-t)}{4}$, so the maximum value is attained at $t = \frac{11}{2}$

$A_{max} = \frac{1}{4}(\frac{11}{2})^2 = \frac{121}{16}$

Answer $= 121 + 16 = 137$

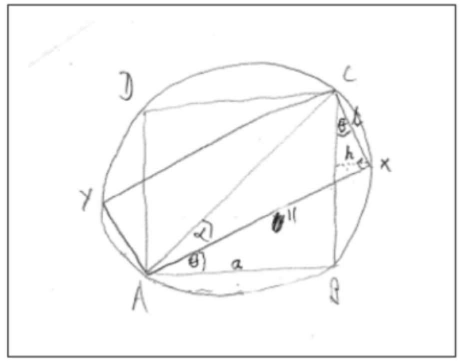


Figure 5: Figure for Q24

---

25. For any positive integer n, let $s(n)$ denote the sum of the digits of n. Find the largest positive integer n such that $n = \left(s(n)\right)^2 + 2s(n) - 2$.
[Answer: 397]

Solution:
Let $n = \sum_{k\geq 0} a_k 10^k, a_k \in \{0, \dots 9\}$. Then $s(n) = \sum_k a_k$.
If n has m digits, then $s(n) \leq 9m$, which gives $s(n)^2 + 2s(n) - 2 \leq 81m^2 + 18m - 2$, while $n \geq 10^{m-1}$. When $m \geq 5$, it's clear there can be no solution.
If $m = 4$, $s(n)^2 + 2s(n) - 2 \leq 81m^2 + 18m - 2 = 1366$, but when $n \leq 1366$, $s(n) \leq 22$, then $s(n)^2 + 2s(n) - 2 \leq 526$ forcing n to have fewer than 4 digits.
We try $m = 3$ next, $s(n)^2 + 2s(n) - 2 \leq 81m^2 + 18m - 2 = 781$, with $n \leq 781$, $s(n) \leq 24$, then $s(n)^2 + 2s(n) - 2 \leq 622$, with $n \leq 622$, $s(n) \leq 23$.
We can proceed with $s(n) = 23$ downwards, at each step calculating $s(n)^2 + 2s(n) - 2$ and to check that if we set $n = s(n)^2 + 2s(n) - 2$, its digits indeed sum up to $s(n)$. The first such occurrence is $s(n) = 19$, with $n = 397$.
