2025 NEAML

SOLUTIONS

Round 1 - Arithmetic and Number Theory

Problem 1. Compute the positive value of the continued fraction $3+\cfrac{2}{3+\cfrac{2}{3+\cfrac{2}{3+\frac{2}{\cdots}}}}$ **in the form** $\frac{A+\sqrt{B}}{C}$ **where** A, B, **and** C **are whole numbers.** ***Note: The pattern continues, with 3's and 2's alternating.***

Solution $\boxed{\mathbf{\frac{3+\sqrt{17}}{2}}}$ **Recognize what the pattern tells us: namely, that the continued fraction is a part of the continued fraction. Let** X **be the value of the continued fraction. Then,** X **is equal to** $3+\frac{2}{X}$**. Solve the equation** $X=3+\frac{2}{X}\rightarrow\frac{2}{X}=X-3$**, so** $2=X^2-3X\rightarrow X^2-3X-2=0\rightarrow X=\frac{3\pm\sqrt{17}}{2}$**. Choose the positive value, which is** $\mathbf{\frac{3+\sqrt{17}}{2}}$**.**

Problem 2. Compute the <u>least</u> value of n **for which** $\frac{n!(n+1)!}{792^3}$ **is an integer perfect square.**

Solution $\boxed{\mathbf{21}}$ **Factoring shows that** $\frac{n!(n+1)!}{792^3}=\frac{(n!)^2}{792^2}\cdot\frac{n+1}{792}$**. The first fraction is a perfect square, and the second is equivalent to** $\frac{n+1}{2^3\cdot 3^2\cdot 11}$**. Notice that this fraction will be a perfect square when** $\frac{n+1}{2\cdot 11}=1$**, which is certainly true when** $n=21$**. Therefore, the answer is 21.**

Problem 3. Let $n=3^{20}7^{25}$**. Compute the number of positive integer divisors of** n^2 **that are less than** n **but do not divide** n**.**

Solution $\boxed{\mathbf{500}}$ **We know that** $n=3^{20}7^{25}$ **has** $21\cdot 26$ **factors. and** $21\cdot 26-1$ **of them are less than** n**.** $n^2=3^{40}7^{50}$ **has** $41\cdot 51=2091\rightarrow 1045$ **pairs of unequal factors whose product is** n^2**. Within each pair one factor must be less than** n**. Thus, there are** $\frac{41\cdot 51-1}{2}=1045$ **factors of** n^2 **that are less than** n**. There are** $21\cdot 26-1=545$ **factors that divide** n **and are less than** n**, so the answer is** $1045-545=$ **500.**

Round 2 - Algebra 1

Problem 1. Given two numbers M **and** N **such that** $M+N=6$ **and** $\frac{1}{M}+\frac{1}{N}=\frac{6}{7}$**, compute** M^2N+MN^2**.**

Solution $\boxed{\mathbf{42}}$ **Combining fractions,** $\frac{1}{M}+\frac{1}{N}=\frac{M+N}{MN}=\frac{6}{7}\rightarrow\frac{6}{MN}=\frac{6}{7}$**, so** $MN=7$**. Now,** $M^2N+MN^2=MN(M+N)=7\cdot 6=$ **42.**

Problem 2. In an arithmetic sequence, the first term is 15 **and the fifth term is** 30**. Compute the thirteenth term.**

Solution $\boxed{60}$ **The difference between the first and fifth terms is the difference between the fifth and ninth terms or the difference between the ninth and thirteenth terms. Thus the answer is** $30 + 2(30 - 15) = \mathbf{60}$**.**

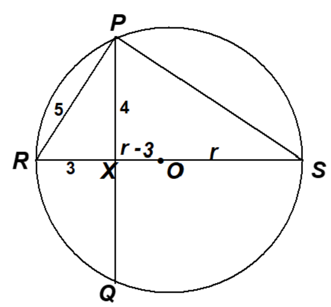
Problem 3. Tim uses two hoses to fill a pool. The blue hose could fill the pool in four hours by itself. The green hose could fill the pool in three hours by itself. Compute the integer nearest the number of minutes it would take these hoses working together to fill the pool.

Solution $\boxed{103}$ **In** x **hours, the blue hose fills** $\frac{x}{4}$ **of a pool and the green hose fills** $\frac{x}{3}$ **of a pool. The goal here is to fill one pool, so solve** $\frac{x}{4} + \frac{x}{3} = 1$ **to obtain** $\frac{7x}{12} = 1 \to x = \frac{12}{7}$ **hours. The number of minutes is** $\frac{12}{7} \cdot 60 \approx \mathbf{103}$**.**

Round 3 - Geometry

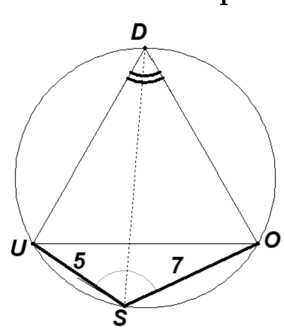
Problem 1. In a circle centered at O**, chord** $\overline{PQ}$ **is perpendicular to diameter** $\overline{RS}$ **at** X**. Given that** $PQ = 8$ **and** $RX = 3$**, compute the radius of the circle.**

Solution $\boxed{\mathbf{\frac{25}{6}}}$ **Because the chord is perpendicular to the diameter,** $PX = QX = 4$**. Let the radius be** r**. Then by Power of a Point,** $4 \cdot 4 = 3 \cdot (2r - 3) \to 6r - 9 = 16 \to r = \mathbf{\frac{25}{6}}$**.**



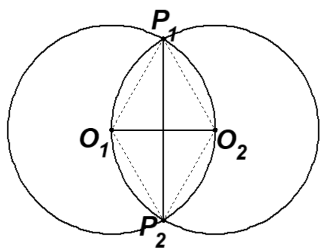
Problem 2. Equilateral triangle DUO **is inscribed in a circle, and** S **is a point on minor arc** UO**. Given that the length of chord** $\overline{US}$ **is** 5 **and the length of chord** $\overline{SO}$ **is** 7**, compute** DS**.**

Solution $\boxed{12}$ **Suppose that** $DU = UO = OD = s$**. Since the marked angles subtend a major and minor arc whose degree measures add up to** $360°$**, quadrilateral** $DUSO$ **is cyclic and Ptolemy's theorem applies, namely the product of the lengths of the opposite sides are equal.** $US \cdot DO + DU \cdot SO = DS \cdot UO$**, so** $5 \cdot s + 7 \cdot s = DS \cdot s$**, so** $DS = \mathbf{12}$**.**



Problem 3. Two circles, centered at O_1 and O_2, each have radius 4, and each passes through the center of the other. Compute the total area enclosed by the circles.

Solution $\boxed{\mathbf{\frac{64\pi}{3}+8\sqrt{3}}}$ **Let the circles intersect at P_1 and P_2. The area enclosed by the circles is equal to the area of the two circles, minus the area of the two sectors $P_1O_1P_2$ and $P_1O_2P_2$, plus the area of quadrilateral $O_1P_1O_2P_2$. The area of the circles is $2\pi(4)^2 = 32\pi$. The area of each sector is $\frac{1}{2}\cdot\frac{2\pi}{3}(4)^2$ for a total of $\frac{32\pi}{3}$. The area of the quadrilateral is the area of a rhombus, which is half the product of the lengths of the diagonals, or $\frac{1}{2}\cdot 4\cdot 4\sqrt{3}$. The answer is $32\pi - \frac{32\pi}{3} + 8\sqrt{3} = \mathbf{\frac{64\pi}{3}+8\sqrt{3}}$.**



Round 4 - Algebra 2

Problem 1. Compute the value of n for which $n! - (n-1)! = 35280$.

Solution $\boxed{8}$ **Factoring obtains $(n-1)!(n-1) = 35280 = 2^4\cdot 3^2\cdot 5\cdot 7^2$. Note that $7! = 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7 = 2^4\cdot 3^2\cdot 5\cdot 7$, so $(n-1) = 7$ and $n = \mathbf{8}$.**

Problem 2. The radical expression $\sqrt{18+\sqrt{35}}$ may be expressed in the form $\frac{x+\sqrt{y}}{\sqrt{z}}$ for positive integers x, y, and z. Compute the ordered triple (x,y,z) for which $x+y+z$ is a minimum.

Solution $\boxed{(\mathbf{1,35,2})}$ **Notice that $(1+\sqrt{35})^2 = 36+2\sqrt{35}$. Dividing by 2 on both sides yields $\left(\frac{1+\sqrt{35}}{\sqrt{2}}\right)^2 = 18+\sqrt{35}$. Taking the square root on both sides tells us that $\sqrt{18+\sqrt{35}} = \frac{1+\sqrt{35}}{\sqrt{2}}$. Our ordered triple is $(\mathbf{1,35,2})$.**

Problem 3. Let the zeroes of $f(x) = 2x^3-3x-1$ be p, q, and r. Compute $\frac{1}{p^2}+\frac{1}{q^2}+\frac{1}{r^2}$.

Solution $\boxed{9}$ **Consider the polynomial $g(x) = x^3\left(\frac{2}{x^3}-\frac{3}{x}-1\right) = -x^3-3x^2+2$. The zeroes of $g(x)$ are $1/p$, $1/q$, and $1/r$. Call these three roots P, Q, and R for convenience. Then, $(P+Q+R)^2 = P^2+Q^2+R^2-2PQ-2QR-2PR$. Thus, $\frac{1}{p^2}+\frac{1}{q^2}+\frac{1}{r^2} = P^2+Q^2+R^2 = (P+Q+R)^2-2(PQ+QR+PR)$, and by Viete's formulas this is $P^2+Q^2+R^2 = (3/-1)^2-2(0) = \mathbf{9}$.**

Round 5 - Analytic Geometry

Problem 1. The circle with equation $x^2+y^2+2x-6y = 12$ has center (h,k) and radius r. Compute $h+k+r^2$.

Solution $\boxed{24}$ **Complete the square to obtain $x^2+2x+1+y^2-6y+9 = 12+1+9 = 22$. This is equivalent to $(x+1)^2+(y-3)^2 = 22$, so $(h,k) = (-1,3)$ and $r^2 = 22$. The answer**

is $-1+3+22=$ **24.**

Problem 2. Rhombus $RHOM$ has vertices at $R(5,5)$, $H(4,1)$, $O(0,0)$, and $M(1,4)$. Compute the area of $RHOM$.

Solution $\boxed{15}$ The area of a rhombus is half the product of the lengths of its diagonals. Notice that $RO = \sqrt{5^2+5^2} = 5\sqrt{2}$ and $HM = \sqrt{3^2+3^2} = 3\sqrt{2}$. Thus, the answer is $[RHOM] = \frac{1}{2}(5\sqrt{2})(3\sqrt{2}) = \mathbf{15}$.

Problem 3. A circle centered at a point on the y-axis above $(0,4)$ is tangent to the lines $y=x$, $y=-x$, $y=4$, and $x=k$ where $k>4$. Compute k.

Solution $\boxed{4\sqrt{2}+4}$ The circle has a center at $C(0,c)$ where $c>4$. Then a radius drawn to the point of tangency with $y=x$ forms a right triangle whose hypotenuse is the segment connecting the origin and C. This right triangle is isosceles because the line $y=x$ is on a 45° angle with the x-axis. Thus $r\sqrt{2} = r+4 \to r = \dfrac{4}{\sqrt{2}-1} = 4\sqrt{2}+4$. By symmetry, vertical lines must be tangent to the circle, so k is this radius. Thus $k = \mathbf{4\sqrt{2}+4}$.

Round 6 - Trig and Complex Numbers

Problem 1. Given that $i=\sqrt{-1}$, compute $\dfrac{625}{(3i-4)^2}$ in simplest $a+bi$ form.

Solution $\boxed{\mathbf{7+24i}}$ The given fraction is equivalent to $\dfrac{625}{-9+16-24i} = \dfrac{625}{7-24i}$. This is equivalent to $\dfrac{(625)(7+24i)}{(7-24i)(7+24i)} = \dfrac{625(7+24i)}{49+576} = \mathbf{7+24i}$.

Problem 2. Equilateral triangle ABC has perimeter 12. Quadrilaterals $ACDE$, $BAFG$, and $CBHI$ are squares constructed on the exterior of $\triangle ABC$. Compute the area of hexagon $DEFGHI$ in the form $A+B\sqrt{C}$.

Solution $\boxed{\mathbf{48+16\sqrt{3}}}$ Notice that $[DEFGHI] = [ACDE]+[BAFG]+[CBHI]+[ABC]+[AEF]+[BGH]+[CDI]$. The area of each square is $4\cdot 4=16$. The area of ABC is $\dfrac{4^2\sqrt{3}}{4}$. The area of each of the other triangles is $\dfrac{1}{2}\cdot 4\cdot 4\cdot GH$, where $GH^2 = 4^2+4^2-2\cdot 4\cdot 4\cdot \cos 120^\circ$ by the Law of Cosines. The answer is $16+16+16+4\sqrt{3}+3\cdot 4\sqrt{3} = \mathbf{48+16\sqrt{3}}$.

Problem 3. The three cube roots of $-4\sqrt{2}+4\sqrt{2}i$ are $A\operatorname{cis}\theta_1$, $A\operatorname{cis}\theta_2$, $A\operatorname{cis}\theta_3$ where $0\le\theta_1\le\theta_2\le\theta_3\le 2\pi$. Compute the ordered pair (A,θ_2).

Solution $\boxed{\left(\mathbf{2}, \dfrac{\mathbf{11\pi}}{\mathbf{12}}\right)}$ Converting to trig form, $-4\sqrt{2}+4\sqrt{2}i = 4\sqrt{2}(-1+i) = r\operatorname{cis}\theta$, where $x^2+y^2=r^2$ and $\tan\theta = \dfrac{y}{x}$.

Thus, $r^2 = 1+1 = 2 \implies r=\sqrt{2}$ and $\tan\theta = \dfrac{1}{-1} = -1$ and θ is in quadrant 2 $\implies \theta = \dfrac{3\pi}{4}$.

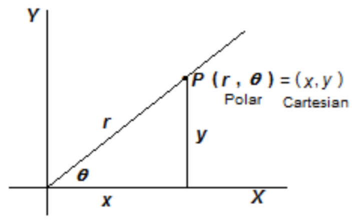
$$-4\sqrt{2}+4\sqrt{2}i = 4\sqrt{2}\cdot\sqrt{2}\operatorname{cis}\left(\frac{3\pi}{4}\right) = 8\operatorname{cis}\left(\frac{3\pi}{4}\right).$$

$$(r\operatorname{cis}\theta)^{1/3} = r^{1/3}\operatorname{cis}\left(\frac{\theta}{3}\right) = 8^{1/3}\operatorname{cis}\frac{1}{3}\left(\frac{3\pi}{4}+2n\pi\right) \implies r=2, \theta = \frac{\pi}{4}+\frac{2n\pi}{3}.$$

$n=0 \implies \theta_1 = \frac{\pi}{4}, n=2 \implies \theta_2 = \frac{\pi}{4} + \frac{2\pi}{3} = \frac{11\pi}{12}$.

The required ordered pair is $\left(2, \frac{11\pi}{12}\right)$**.**

The relationship between rectangular coordinates and polar coordinates is illustrated below.

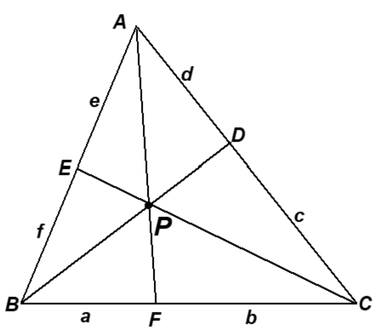


NEAML TEAM ROUND

Problem 1. Compute the term in the expansion of $\left(2x^3 + \frac{1}{x^7}\right)^{10}$ **that contains no** x**'s.**

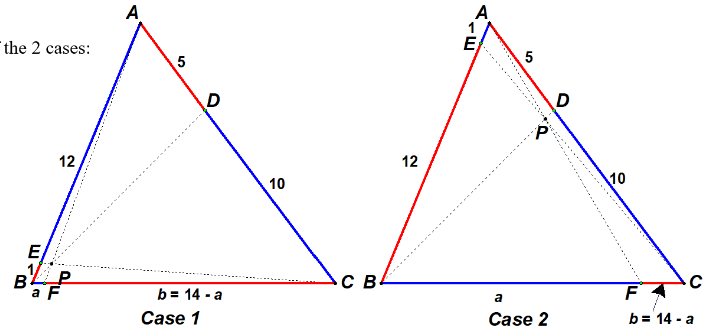
Solution $\boxed{15360}$ **The numerical coefficient is** $\binom{10}{k}2^{10-k}$ **where the value of** k **is determined by setting the literal component equal to** x^0**. Thus,** $x^{3(10-k)} \cdot x^{-7k} = x^{30-10k} = x^0 \to 10k = 30 \to k = 3$. **Therefore, the numerical coefficient is** $\binom{10}{3} \cdot 2^7 = \frac{10!}{3! \cdot 7!} \cdot 2^7$**=15360.**

Problem 2. ΔABC **has sides of lengths 13, 14, and 15. A cevian is drawn to a trisection point of the longest side. A second cevian is drawn dividing the shortest side into segments in a** $12:1$ **ratio. A third cevian passes through the point of intersection of the first two cevians, and divides the third side of the triangle into two segments. Compute the shortest possible length of one of the six segments.**



Solution $\boxed{\frac{14}{25}}$ **Applying Ceva's theorem, we know** $ace = bdf$ **for any three cevians intersecting in a common point in the interior of a triangle. In this question, we know 4 lengths of the segments on the sides of** ΔABC**, but not their relative locations (5 and 10 on the longest side, and 1 and 12 on the shortest side). Is there a segment on whose length is less than 1? Case 1:** $a \cdot 10 \cdot 12 = b \cdot 5 \cdot 1 \Leftrightarrow 120a = 5b \Leftrightarrow 24a = b \Rightarrow BC = 25a = 14 \Rightarrow a = \frac{14}{25}$**. Case 2:** $a \cdot 10 \cdot 1 = b \cdot 5 \cdot 12 \Leftrightarrow 10a = 60b \Rightarrow a = 6b \Rightarrow BC = 7b = 14 \Rightarrow b = 2$**. If the positions of the segments of lengths 5 and 10 are reversed, there is no change (other than the roles of** a **and** b **are reversed, and case 1 becomes case 2, and vice versa). The smallest possible length of any of the 6 segments is** $\frac{14}{25} = 0.56$**.**

Diagrams of the 2 cases:



Problem 3. **The height of water in a reservoir is given by** $H(t) = 20 \cdot \cos\left(\frac{2\pi}{365}(t-80)\right) + 100$ **where** t **is measured in days after April 1 and** H **is measured in feet. Given that** M **is the maximum height of the water over a calendar year in feet and** P **is the length of time in days between maximum heights, compute** $M + P$**.**

Solution $\boxed{485}$ **Notice that** $M = 100 + 20 = 120$ **because** 20 **is the amplitude of the wave and** 100 **is the midline. Also,** $P = \dfrac{2\pi}{(2\pi/365)} = 365$ **and so it takes** 365 **days to return to its maximum. Thus the answer is** $M + P = 120 + 365 = \mathbf{485}$**.**

Problem 4. **For the sequence** a_n**, suppose that** $a_1 = 20$, $a_2 = 19$**, and** $a_n = a_{n-1} + a_{n-2}$ **for** $n \geq 3$**. Of the first 2019 terms, compute the number of terms that are divisible by 3.**

Solution $\boxed{505}$ **Consider the remainders when dividing by 3. The first two are 2 and 1, and then using the rule we obtain 0, 1, 1, 2, 0, 2, 2, 1, ... The cycle has length 8 (2, 1, 0, 1, 1, 2, 0, 2), and there are** $2016 \div 8 = 252$ **cycles in the first 2016 terms. The number of the first 2019 terms that is divisible by 3 is** $252 \cdot 2 + 1 = \mathbf{505}$**.**

Problem 5. **Three positive integers** a, b**, and** c**, with** $a < b < c$**, satisfy** $a + b + c = 40$**. The positive pairwise differences of these numbers are** 5, 9**, and** 14**. Compute** $a^2 + c^2$**.**

Solution $\boxed{490}$ **Because the biggest of the pairwise differences is** 14**, it follows that** $c - a = 14$**. It is true either that** $c - b = 5$ **and** $b - a = 9$ **or** $c - b = 9$ **and** $b - a = 5$**.**
Assume that $c - b = 5$ **and** $b - a = 9$**. Then** $(b - a) + (a + b + c) = 9 + 40 \rightarrow 2b + c = 49$**. This implies that** $(2b + c) - (c - b) = 49 - 5 \rightarrow 3b = 44$**, which is not an integer. This is a contradiction.**
Assume that $c - b = 9$ **and** $b - a = 5$**. Then** $(b - a) + (a + b + c) = 5 + 40 \rightarrow 2b + c = 45$**. This implies that** $(2b + c) - (c - b) = 45 - 9 \rightarrow 3b = 36 \rightarrow b = 12$**. It follows that** $a = 7$ **and** $c = 21$**. The answer is** $7^2 + 21^2 = 49 + 441 = \mathbf{490}$**.**

Problem 6. **Consider the sequence** $2, a, b, 8$**. The first three terms are in a harmonic sequence (where the reciprocal of the middle term is equal to the average of the reciprocals of the other two terms). The last three terms are in an increasing arithmetic sequence. Given that** b **is** $A + B\sqrt{C}$ **when written in simplest form, compute** $A^2 + B^2 + C^2$

Solution $\boxed{17}$ **Because the last three terms are in arithmetic sequence,** $b - a = 8 - b \rightarrow a = 2b - 8$**. Now, apply the definition of harmonic sequence:** $\dfrac{1}{2b-8} = \dfrac{\frac{1}{2} + \frac{1}{b}}{2}$ **implies** $2 = \dfrac{2b-8}{2} + \dfrac{2b-8}{b}$**. Multiplying through by** $2b$ **yields** $4b = 2b^2 - 8b + 4b - 16$**, or** $0 = b^2 - 4b - 8$**, which solves to obtain only one positive solution (necessary for an increasing arithmetic sequence):** $x = 2 + 2\sqrt{3}$**. Thus the answer is** $2^2 + 2^2 + 3^2 = 4 + 4 + 9 = \mathbf{17}$**.**