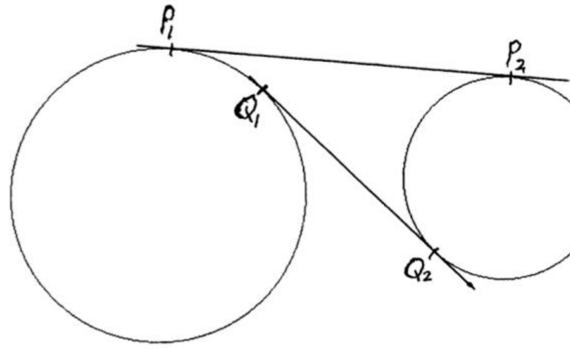
CT ARML Team, 2025
Team Selection Test 1

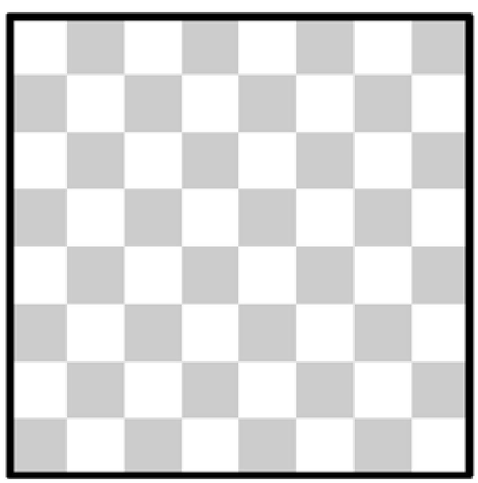
1. Compute the sum of all values of x such that $\log_2(\log_4 x) + \log_4(\log_2 x) = \frac{7}{2}$.

2. Two non-intersecting circles are given. Lines $\overline{P_1P_2}$ and $\overline{Q_1Q_2}$ are tangent to the circles with points of tangency as shown in the diagram below. Given that $P_1P_2 = 11$ and $Q_1Q_2 = 7$, compute the product of the radii of the two circles.

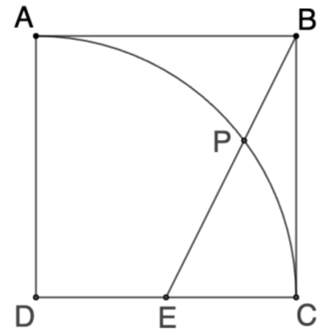


3. Compute the least positive real number k such that the graphs of $y = kx$ and $y = x \cdot \lfloor x \rfloor$ have an intersection of length greater than 10.
[Notation: $\lfloor x \rfloor$, called the *floor* of x, denotes the greatest integer less than or equal to x.]

4. Consider the set of rectangles formed by the unit squares of an 8×8 checkerboard, as shown in the diagram below. Compute the number of these rectangles that contain exactly four black squares.



5. In the diagram below, $ABCD$ is a square of side-length 10, E is the midpoint of side $\overline{CD}$, and arc $\widehat{APC}$ is a quarter-circle centered at D. BPE is a straight line. Find $(BP)^2$.



6. The positive solution of the equation $4x^3 - 3x - \frac{\sqrt{2}}{2} = 0$ is $a\sqrt{6} + b\sqrt{2}$, where a and b are rational numbers. Find $\frac{1}{ab}$.

7. The sequence of positive terms $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 1$ and, for $n \geq 1$, $a_{n+1}^2 = a_n^2 + (n+1)^3$. If we let $S = \frac{1}{a_1} + \cdots + \frac{1}{a_{99}}$, then $S = \frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

8. Find the sum of the real solutions of the equation $x^4 - 8x^3 + 16x^2 - 81 = 0$.

9. Suppose that point D lies on side $\overline{AC}$ of triangle ABC. Given that $\text{m}\angle BAC = 20°$, $\text{m}\angle ABD = 10°$, $\text{m}\angle BCA = 80°$, and $AD = 12$, compute the length BC.

10. Isabel and Robert play a game in which Isabel chooses a list of six real numbers $(x_1, x_2, \ldots, x_6)$, and Robert devises six questions, each of the form "What is $x_j + x_k$", where $1 \leq j < k \leq 6$. Without regard to the order in which Robert asks the questions, in how many ways can Robert choose his questions so that he can determine Isabel's numbers?

CT ARML Team, 2025
Team Selection Test 1
Answers

1. 256
2. 18
3. 10
4. 120
5. 20
6. 16
7. 149
8. 4
9. 12
10. 2530

CT ARML Team, 2025
Team Selection Test 1

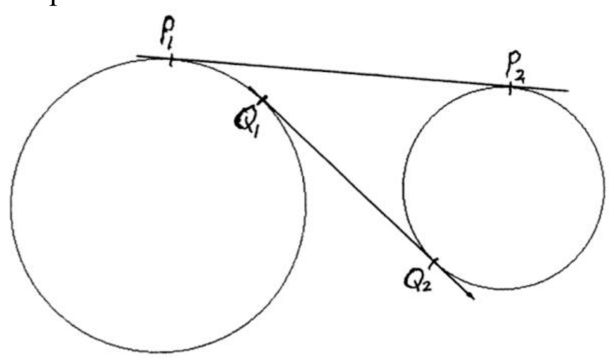
1. Compute the sum of all values of x such that $\log_2(\log_4 x) + \log_4(\log_2 x) = \frac{7}{2}$.

Solution:
Let $\log_4 x = a$, then $\log_2 x = 2a$. We substitute these to the original equation and get

$$\log_2 a + \log_4(2a) = \frac{7}{2}$$
$$\log_2 a + \log_4 a + \frac{1}{2} = \frac{7}{2}$$
$$3\log_4 a = 3$$

So $a = 4, and\ x = 4^a = 256$. The answer is $\boxed{256}$.

2. Two non-intersecting circles are given. Lines $\overline{P_1P_2}$ and $\overline{Q_1Q_2}$ are tangent to the circles with points of tangency as shown in the diagram below. Given that $P_1P_2 = 11$ and $Q_1Q_2 = 7$, compute the product of the radii of the two circles.



Solution. If d be the distance between the centers and r_1 and r_2 be the radii of the circles, WLOG, we can assume $r_1 \geq r_2$. We have, by pythagoras, $d^2 = 11^2 + (r_1 - r_2)^2$, and also, $d^2 = 7^2 + (r_1 + r_2)^2$

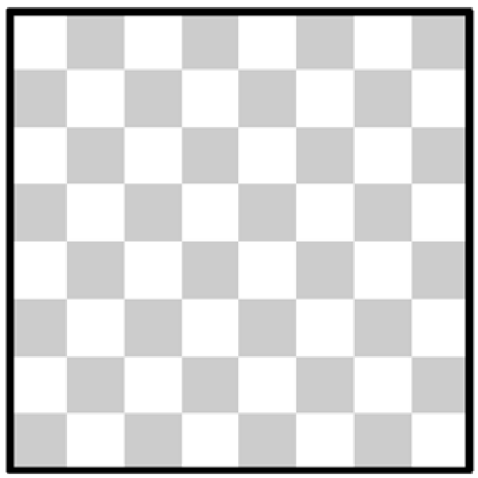
Subtracting, $7^2 + (r_1 + r_2)^2 = 11^2 + (r_1 - r_2)^2$.

Rearranging, $4r_1r_2 = 11^2 - 7^2 = (11 - 7)(11 + 7)$, or, $r_1r_2 = 11 + 7 = 18$

3. Compute the least positive real number k such that the graphs of $y = kx$ and $y = x \cdot \lfloor x \rfloor$ have an intersection of length greater than 10.
[Notation: $\lfloor x \rfloor$, called the *floor* of *x*, denotes the greatest integer less than or equal to *x*.]

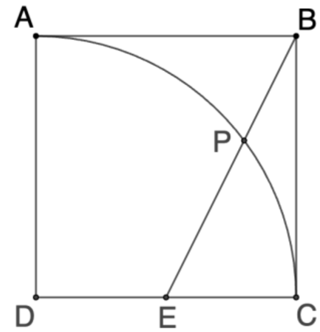
Solution. Only a positive integer k ensures a positive length intersection, and when k is such a positive integer, the intersection happens on the segment $k \leq x < (k+1)$, and the length is $\sqrt{k^2+1}$. Thus we have to solve $\sqrt{k^2+1} \geq 10$. Or $k^2 \geq 99$.
The least positive integral value for k is 10.

4. Consider the set of rectangles formed by the unit squares of an 8×8 checkerboard, as shown in the diagram below. Compute the number of these rectangles that contain exactly four black squares.



<u>Solution</u>:
We can use casework counting to solve this problem.
Case 1: 1×7 or 7×1 rectangles with four black squares. We have 16 of them (one each at each row or column).
Case 2: 1×8 or 8×1 rectangles with four black squares. We have 16 of them (one each at each row or column).
Case 3: 2×4 or 4×2 rectangles. For each subcase, we have 5×7=35 ways to choose the rectangle and a total of 70 ways.
Case 4: 3×3 squares with white corners. We have 6×3=18 ways to choose the square.
Answer: 16 + 16 + 70 + 18 = $\boxed{120}$

5. In the diagram below, $ABCD$ is a square of side-length 10, E is the midpoint of side $\overline{CD}$, and arc $\widehat{APC}$ is a quarter-circle centered at D. BPE is a straight line. Find $(BP)^2$.



Solution. Extend $\overline{BE}$ to intersect $\overline{AD}$ at F. Then $AD = DF$ and F is also on the circle APC and diametrically opposite to A.

$$BF = \sqrt{20^2 + 10^2} = 10\sqrt{5}$$

By the power of the point B at this circle, $BPBF = BA^2$, or

$$\begin{aligned} BP &= \frac{100}{10\sqrt{5}} \\ &= \frac{10}{\sqrt{5}} \end{aligned}$$

Therefore $BP^2 = 20$

6. The positive solution of the equation $4x^3 - 3x - \frac{\sqrt{2}}{2} = 0$ is $a\sqrt{6} + b\sqrt{2}$, where a and b are rational numbers. Find $\frac{1}{ab}$.

Solution:

Let $f(x) = 4x^3 - 3x - \frac{\sqrt{2}}{2}$. Since $f(0) < 0, f\left(-\frac{1}{2}\right) > 0$, we know that there are one positive and two negative solutions.

Substituting $x = a\sqrt{6} + b\sqrt{2}$ into the equation, comparing the coefficients of the $\sqrt{6}$ and $\sqrt{2}$ terms, we get:

$$\begin{cases} 24a^2 + 24b^2 - 3 = 0, \\ 72a^2b + 8b^3 - 3b = \dfrac{1}{2}. \end{cases}$$

Eliminating a, we get:

$$128b^3 - 12b + 1 = 0.$$

This equation can be solved by factoring:

$$128b^3 - 32b^2 + 32b^2 - 8b - 4b + 1 = 0,$$

$$32b^2(4b-1) + 8b(4b-1) - (4b-1) = 0,$$

$$(32b^2 + 8b - 1)(4b - 1) = 0,$$

$b = \frac{1}{4}$ is the only rational solution.

From this we get $a = \frac{1}{4}$. Therefore $\frac{1}{ab} = 16$.

Alternatively:

From the trigonometric identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, and the fact that $\frac{\sqrt{2}}{2} = \cos 45°$, we see that $x = \cos 15° = \cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30° = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$.

7. The sequence of positive terms $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 1$ and, for $n \geq 1$, $a_{n+1}^2 = a_n^2 + (n+1)^3$. If we let $S = \frac{1}{a_1} + \cdots + \frac{1}{a_{99}}$, then $S = \frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

Solution:
By induction, we have:

$$a_n = \frac{n(n+1)}{2},$$

$$\frac{1}{a_n} = \frac{2}{n(n+1)} = \frac{2}{n} - \frac{2}{n+1}.$$

Therefore $x = 2 \cdot \left(1 - \frac{1}{100}\right) = \frac{99}{50}$, $p + q = 149$.

8. Find the sum of the real solutions of the equation $x^4 - 8x^3 + 16x^2 - 81 = 0$.

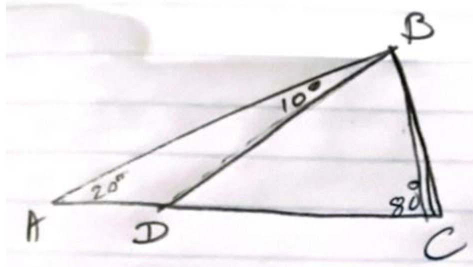
Solution:

$$x^4 - 8x^3 + 16x^2 - 81 = (x^2 - 4x)^2 - 9^2 = (x^2 - 4x + 9)(x^2 - 4x - 9).$$

The first factor produces two conjugate complex solutions, while the second factor produces two real solutions. Their sum is therefore 4.

9. Suppose that point D lies on side $\overline{AC}$ of triangle ABC. Given that $\mathrm{m}\angle BAC = 20°$, $\mathrm{m}\angle ABD = 10°$, $\mathrm{m}\angle BCA = 80°$, and $AD = 12$, compute the length BC.

Solution. First approach is to use trig. Using laws of sine in ABD and BDC, show that $BC = AD$.



$$\frac{AD}{\sin 10°} = \frac{BD}{\sin 20°}$$
$$\frac{BD}{\sin 80°} = \frac{BC}{\sin 30°}$$

Using the doubling formula for sine, and $\sin 80° = \cos 10°$, and $\sin 30° = 1/2$, we get the result.

There is a geometric method as well

10. Isabel and Robert play a game in which Isabel chooses a list of six real numbers $(x_1, x_2, \ldots, x_6)$, and Robert devises six questions, each of the form "What is $x_j + x_k$", where $1 \le j < k \le 6$. Without regard to the order in which Robert asks the questions, in how many ways can Robert choose his questions so that he can determine Isabel's numbers?

Solution:

We can consider the game as a graph with 6 vertices, representing Isabel's numbers, and 6 edges, representing Robert's questions. Note that we can't have a cycle with even length since then the corresponding system of equations wouldn't be linearly independent. For example, the system of equations $x + y = c_1$; $y + z = c_2$; $z + w = c_3$; $w + x = c_4$ is not linearly independent, since the sum of the 1st and 3rd equations is equal to the sum of the 2nd and 4th; hence one equation is wasted, yet 6 independent equations are required to find 6 variables. Since the largest tree in a graph with 6 vertices consists of 5 edges, a cycle must exist. We distinguish three cases, depending on the length of the largest cycle:

Case 1: we have a 5-cycle. There are 6 ways to choose which number is left out of the 5-cycle, 12 ways to order the elements of the 5-cycle, and 5 ways to connect the number left out to the cycle, for a total of 360 possibilities.

Case 2: there are two 3-cycles. As the cycles are disjoint, there are $\frac{\binom{6}{3}}{2} = 10$ such possibilities.

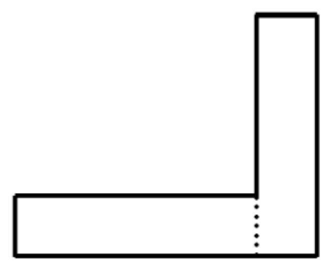
<u>Case 3</u>: there is only one 3-cycle. We choose the cycle C in $\binom{6}{3} = 20$ ways, and then choose the paths to connect the other three vertices X, Y, Z to C; where each of these points must have exactly 1 path. If X →C, Y → C and Z → C; there are $3^3 = 27$ ways; if X → Y → C and Z → C (and permutations), there are $6 \cdot 3^2 = 54$ ways; if X →Z; Y → Z and Z → C (and permutations), there are 3 × 3 = 9 ways; Finally, if X → Y → Z → C (and permutations), there are 6×3 = 18 ways, for a total of 20× (27+54+9+18) = 2160 possibilities.

Thus, Robert has a total of 360 + 10 + 2160 = $\boxed{2530}$ ways to ask Isabel his questions.

Problem 1A-1. Compute the number of eight-digit positive integers whose digits from left to right are strictly decreasing.

Problem 1A-2. *Let T be the number you will receive.* Compute the number of digits in the decimal expansion of 20^T.

Problem 1A-3. *Let T be the number you will receive.* The figure below results from adjoining two $a \times b$ rectangles, where a and b are positive integers, $a < b$, and one of the smaller sides of one rectangle fully overlaps with a larger side of the other rectangle. Given that the perimeter of the figure is $2T$, compute the number of possible ordered pairs (a, b).



Answer 1A-1. 45

Answer 1A-2. 59

Answer 1A-3. 10

11 Collaborative Relay Solutions

Problem 1A-1. Compute the number of eight-digit positive integers whose digits from left to right are strictly decreasing.

Solution 1A-1. For any integer N satisfying the conditions of the problem, exactly two of the ten digits $0, 1, \ldots, 9$ do *not* appear in N. There are $\binom{10}{2} = 45$ ways to choose the two digits that do not appear in N, and there is exactly one way to order the remaining eight digits to form an integer whose digits are strictly decreasing from left to right. Hence the answer is **45**.

Alternate Solution: Because any integer satisfying the conditions of the problem has eight digits, the leftmost digit must be greater than 6. Now consider three cases.

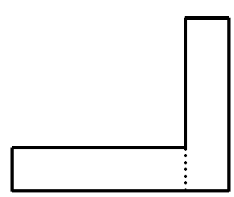
- If the leftmost digit is 7, there is only one possibility: 76543210.
- If the leftmost digit is 8, then the rightmost seven digits are seven of the eight digits 7, 6, 5, 4, 3, 2, 1, 0. There are 8 ways to choose which of these eight digits is *not* used, and then there is exactly one way to order the remaining seven digits to form an integer whose digits are strictly decreasing from left to right.
- If the leftmost digit is 9, then the rightmost seven digits are seven of the nine digits 8, 7, 6, 5, 4, 3, 2, 1, 0. There are $\binom{9}{2} = 36$ ways to choose the two digits that are *not* used, and then there is exactly one way to order the remaining seven digits to form an integer whose digits are strictly decreasing from left to right.

Thus the answer is $1 + 8 + 36 = 45$.

Problem 1A-2. *Let T be the number you will receive.* Compute the number of digits in the decimal expansion of 20^T.

Solution 1A-2. Notice that $2^{10} = 1024 \approx 10^3$ so $2^{45} = 2^5 \cdot (2^{10})^4 \approx 32 \cdot 10^{12}$, which has 14 digits. Multiplying by 10^{45} simply appends 45 zeroes, so 20^{45} has $14 + 45 = \mathbf{59}$ digits.

Problem 1A-3. *Let T be the number you will receive.* The figure below results from adjoining two $a \times b$ rectangles, where a and b are positive integers, $a < b$, and one of the smaller sides of one rectangle fully overlaps with a larger side of the other rectangle. Given that the perimeter of the figure is $2T$, compute the number of possible ordered pairs (a, b).



Solution 1A-3. The perimeter of the figure is $2(a + b) + 2b = 2a + 4b$. Hence $2a + 4b = 2T$, so $a + 2b = T$. From $a < b$, it follows that $T = a + 2b < 3b$, so $b > T/3$. Moreover, if T is odd, then the greatest possible value of b is $\lfloor T/2 \rfloor$, and if T is even, then the greatest possible value of b is $\frac{T-2}{2}$. Hence the greatest possible value of b is $\lceil T/2 \rceil - 1$. Given a fixed value of b, the value of a is uniquely determined and the number of integral values of b in the interval $(T/3, \lceil T/2 \rceil - 1]$ is $\lceil T/2 \rceil - 1 - (\lfloor T/3 \rfloor + 1) + 1 = \lceil T/2 \rceil - \lfloor T/3 \rfloor - 1$. With $T = 59$, the answer is $30 - 19 - 1 = \mathbf{10}$.

1 Team Problems

Problem 1. Let N be the 20-digit number 20202020202020202020. Compute the sum of the digits of N^2.

Problem 2. In a regular 20-gon, three distinct vertices are chosen at random. Compute the probability that the triangle formed by these three vertices is a right triangle.

Problem 3. Starting at $(0, 0)$, a frog moves in the coordinate plane via a sequence of hops. Each hop is either 1 unit in the x-direction or 1 unit in the y-direction. Compute the minimum number of hops needed for the frog to land on the line $15x + 35y = 2020$.

Problem 4. Compute the number of real values of x such that $\cos(\cos(x)) = \dfrac{x}{10}$.

Problem 5. A circle passes through both trisection points of side $\overline{AB}$ of square $ABCD$ and intersects $\overline{BC}$ at points P and Q. Compute the greatest possible value of $\tan \angle PAQ$.

Problem 6. Compute the least integer $n > 2020$ such that $(n + 2020)^{n-2020}$ divides n^n.

Problem 7. The *latus rectum* of a parabola is the line segment parallel to the directrix, with endpoints on the parabola, that passes through the focus. Define the *latus nextum* of a parabola to be the line segment parallel to the directrix, with endpoints on the parabola, that is twice as long as the latus rectum. Compute the greatest possible distance between a point on the latus rectum and a point on the latus nextum of the parabola whose equation is $y = x^2 + 20x + 20$.

Problem 8. Circle Ω_1 with radius 11 and circle Ω_2 with radius 5 are externally tangent. Circle Γ is internally tangent to both Ω_1 and Ω_2, and the centers of all three circles are collinear. Line ℓ is tangent to Ω_1 and Ω_2 at distinct points D and E, respectively. Point F lies on Γ so that $FD < FE$ and $m\angle DFE = 90°$. Compute $\sin \angle DEF$.

Problem 9. Nine distinct circles are drawn in the plane so that at most half of the pairs of these circles intersect. The circles divide the plane into N regions of finite area. Compute the maximum possible value of N.

Problem 10. For a positive integer k, let $S(k)$ denote the sum of the digits of k. Compute the number of seven-digit positive integers n such that $S(n) + 4S(2n) = S(36n)$.

2 Answers to Team Problems

Answer 1. 112

Answer 2. $\frac{3}{19}$

Answer 3. 60

Answer 4. 3

Answer 5. $\frac{7}{11}$

Answer 6. 2076

Answer 7. $\frac{3\sqrt{5}}{4}$

Answer 8. $\frac{\sqrt{5}}{4}$

Answer 9. 39

Answer 10. 610

3 Solutions to Team Problems

Problem 1. Let N be the 20-digit number 20202020202020202020. Compute the sum of the digits of N^2.

Solution 1. Let $M = 1010101010101010101$. Note that $N = 20M$ and that

$$M^2 = 1020304050607080910090807060504030201.$$

Squaring N,

$$N^2 = 400M^2 = 408121620242832364036322824201612080400.$$

The sum of the digits of N^2 is thus 4 more than twice the sum of all the digits of the positive multiples of 4 up to 36, which is

$$2(4+8+1+2+1+6+2+0+2+4+2+8+3+2+3+6)+4 = \mathbf{112}.$$

Problem 2. In a regular 20-gon, three distinct vertices are chosen at random. Compute the probability that the triangle formed by these three vertices is a right triangle.

Solution 2. There are $\binom{20}{3} = 1140$ triangles that can be formed by choosing three distinct vertices of the 20-gon. Inscribe the 20-gon in a circle. By the Inscribed Angle Theorem, a right triangle is only formed in the case in which two of the vertices chosen (at random) are diametrically opposite each other. There are 10 choices for a diameter; after the diameter is chosen, the third vertex can be chosen in 18 ways. Thus the answer is $\dfrac{18\cdot 10}{1140} = \mathbf{\dfrac{3}{19}}$.

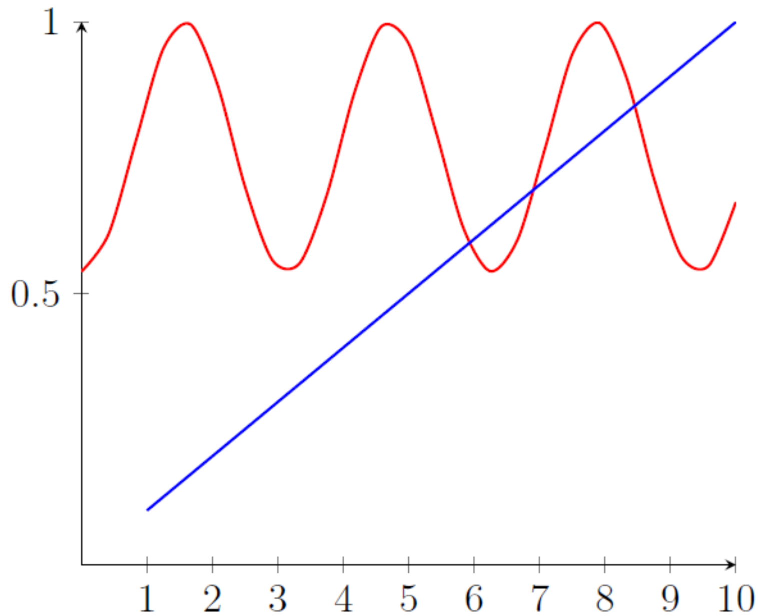
Problem 3. Starting at $(0,0)$, a frog moves in the coordinate plane via a sequence of hops. Each hop is either 1 unit in the x-direction or 1 unit in the y-direction. Compute the minimum number of hops needed for the frog to land on the line $15x + 35y = 2020$.

Solution 3. The frog can only land on lattice points. Let (a,b) denote the point on the line $15x+35y = 2020$ on which the frog lands. The frog lands on the line only if $15a + 35b = 2020$, and the number of hops required is $|a| + |b|$. Note that $15a + 35b = 2020$ implies $3a + 7b = 404$. This line passes through the points $(134\frac{2}{3}, 0)$ and $(0, 57\frac{5}{7})$.

Consider lines of the form $x + y = k$. Because $x + y = k$ has a slope of -1 and $15x + 35y = 2020$ has a slope of $-\frac{15}{35} = -\frac{3}{7}$, the lattice point that produces the minimum number of hops must be as close as possible to the y-intercept $(0, 57\frac{5}{7})$. Notice that if $x = 1$ or $x = 2$ or $x = 3$, then there is no integer solution to $15x + 35y = 2020$ with respect to y. If $x = 4$, then $15\cdot 4 + 35y = 2020 \to y = 56$, which is an integer. Thus the minimum number of hops needed for the frog to land on the line $15x + 35y = 2020$ is $4 + 56 = \mathbf{60}$.

Problem 4. Compute the number of real values of x such that $\cos(\cos(x)) = \dfrac{x}{10}$.

Solution 4. Let $f(x) = \cos(\cos(x))$ and $g(x) = \frac{x}{10}$. The function $\cos(x)$ is even and ranges from -1 to 1, and so $f(x)$ ranges between $\cos(1)$ and $\cos(0) = 1$ with period π and maxima and minima occurring at $\frac{k\pi}{2}$ for odd and even integers k, respectively. The graphs of $f(x)$ and $g(x)$ are shown below.



The ranges of $f(x)$ and $g(x)$ are identical for $x \in [10\cos(1), 10]$, with $g(x)$ increasing over the entirety of this interval. Note that

$$\cos(1) = \cos\left(\frac{\pi}{3} - \left(\frac{\pi}{3} - 1\right)\right) = \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3} - 1\right) + \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3} - 1\right)$$
$$< \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{\pi}{3} - 1\right) < \frac{1}{2} + \frac{1}{21} < \frac{3}{5} < \frac{\pi}{5},$$

with the latter inequalities using the facts that $\cos(x) \leq 1$ and $\sin(x) < x$ for all positive x, $\frac{\sqrt{3}}{2} < 1$, and $3 < \pi < \frac{22}{7}$. Thus it follows that $g(10\cos(1)) = \cos(1) < f(10\cos(1))$ and $f(2\pi) = \cos(1) < \frac{\pi}{5} = g(2\pi)$. Because $f(x)$ is strictly decreasing between consecutive maxima and minima, there is a single intersection point of the two curves in the interval $(10\cos(1), 2\pi)$. By a similar argument, it follows that there is an intersection point of the two curves in each of the intervals $(2\pi, \frac{5\pi}{2})$ and $(\frac{5\pi}{2}, 3\pi)$.

There is no intersection point of the curves in the interval $(3\pi, 10)$ because

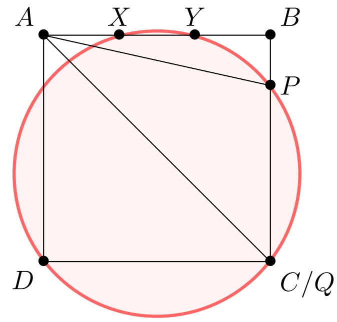
$$\begin{aligned} f(x) < f\Big(3\pi + \text{Arccos}\left(\frac{\pi}{6}\right)\Big) &= \cos\Big(\cos\Big(3\pi + \text{Arccos}\left(\frac{\pi}{6}\right)\Big)\Big) \\ &= \cos\Big(\cos(3\pi)\cos\Big(\text{Arccos}\left(\frac{\pi}{6}\right)\Big) - \sin(3\pi)\sin\Big(\text{Arccos}\left(\frac{\pi}{6}\right)\Big)\Big) \\ &= \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} < g(3\pi) \end{aligned}$$

for all x in the interval, as $10 < 3\pi + \text{Arccos}(\frac{\pi}{6}) < \frac{7\pi}{2}$.

Thus there are a total of **3** intersection points.

Problem 5. A circle passes through both trisection points of side $\overline{AB}$ of square $ABCD$ and intersects $\overline{BC}$ at points P and Q. Compute the greatest possible value of $\tan\angle PAQ$.

Solution 5. Suppose, without loss of generality, that $ABCD$ has sides of length 1; denote the trisection points of $\overline{AB}$ by X and Y so that $AX = XY = YB = \frac{1}{3}$. Let P be nearer to B and let Q be nearer to C.



By Power of a Point from B, $BP \cdot BQ = BY \cdot BX = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$; thus if $BQ = x$, then $BP = \frac{2}{9x}$. Because $\overline{BC}$ intersects the circle in points P and Q and P is between B and Q, it follows that $BP < BQ \leq BC = 1$, so $\frac{2}{9x} < x \leq 1$. Note that maximizing $\tan \angle PAQ$ is equivalent to maximizing PQ. Geometrically, this value is maximized when Q coincides with C, so $x = 1$. This also agrees algebraically: to maximize $PQ = x - \frac{2}{9x}$, x should be as large as possible and $\frac{2}{9x}$ should be as small as possible. Given the bounds on x, the expression $x - \frac{2}{9x}$ achieves its maximum when $x = 1$. This implies

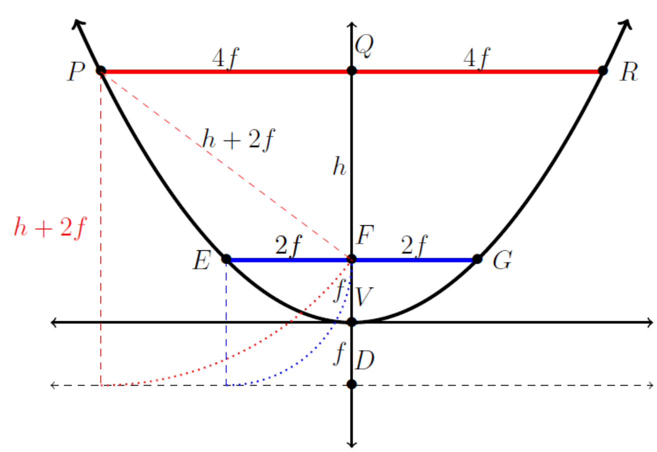
$$\tan \angle PAQ = \tan(\angle QAB - \angle PAB) = \frac{x - \frac{2}{9x}}{1 + \frac{2}{9}} = \mathbf{\frac{7}{11}}.$$

Problem 6. Compute the least integer $n > 2020$ such that $(n + 2020)^{n-2020}$ divides n^n.

Solution 6. Let $m = n + 2020 > 4040$. Suppose a prime p divides m. Then p also divides n, and therefore $p \mid 2020 = 2^2 \cdot 5 \cdot 101$. In other words, m is an integer greater than 4040 with no prime factors other than 2, 5, and 101. Note the least candidate for m is $4096 = 2^{12}$; indeed, the next multiple of 101 after 4040 is 4141, and there is no number of the form $2^a 5^b$ between 4040 and 4096. So the least candidate for n is 2076. This works, because $(n + 2020)^{n-2020} = 4096^{56} = 2^{12 \cdot 56}$ certainly divides 2076^{2076}. Thus the answer is **2076**.

Problem 7. The *latus rectum* of a parabola is the line segment parallel to the directrix, with endpoints on the parabola, that passes through the focus. Define the *latus nextum* of a parabola to be the line segment parallel to the directrix, with endpoints on the parabola, that is twice as long as the latus rectum. Compute the greatest possible distance between a point on the latus rectum and a point on the latus nextum of the parabola whose equation is $y = x^2 + 20x + 20$.

Solution 7. Denote by h the distance between the latus rectum and the latus nextum. Let f denote the focal length (so the latus rectum has length $4f$). The diagram below reveals the critical relation $(h + 2f)^2 = h^2 + (4f)^2 \to h = 3f$.



Calculate either the length ER or PG shown in the diagram to find that the greatest possible distance is

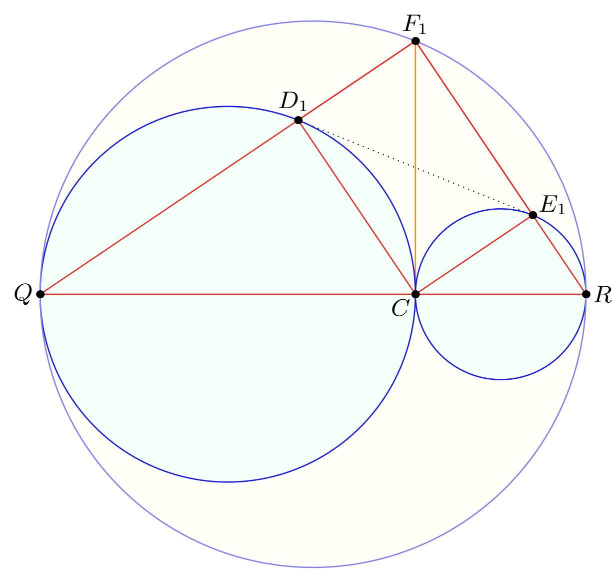
$$\sqrt{(3f)^2 + (6f)^2} = 3\sqrt{5}f.$$

Recall that for a general parabola $y = ax^2 + bx + c$, the focal length is $f = \dfrac{1}{4a}$. In particular, for this parabola, the focal length is $f = \dfrac{1}{4}$, and thus the final answer is $\boldsymbol{\dfrac{3\sqrt{5}}{4}}$.

Alternate Solution: Because the parabola $y = x^2 + 20x + 20$ is congruent to the parabola $y = x^2$, consider instead the parabola $y = x^2$. The latus rectum is along $y = \frac{1}{4}$, and its endpoints are $\left(\pm\frac{1}{2}, \frac{1}{4}\right)$. Thus the latus nextum has endpoints at $(\pm 1, 1)$. The answer is thus $\sqrt{\left(1 - (-\frac{1}{2})\right)^2 + \left(1 - \frac{1}{4}\right)^2} = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3}{4}\right)^2} = \frac{3\sqrt{5}}{4}$.

Problem 8. Circle Ω_1 with radius 11 and circle Ω_2 with radius 5 are externally tangent. Circle Γ is internally tangent to both Ω_1 and Ω_2, and the centers of all three circles are collinear. Line ℓ is tangent to Ω_1 and Ω_2 at distinct points D and E, respectively. Point F lies on Γ so that $FD < FE$ and $\mathrm{m}\angle DFE = 90°$. Compute $\sin\angle DEF$.

Solution 8. Let C be the tangency point of Ω_1 and Ω_2, and construct diameters $\overline{QC}$ and $\overline{CR}$ of Ω_1 and Ω_2, as shown. Let F_1 be the point on Γ such that $\overline{CF_1} \perp \overline{QR}$. Let segment QF_1 intersect Ω_1 at D_1, and let segment RF_1 intersect Ω_2 at E_1. It is not difficult to see that the following are similar right triangles: CQD_1, F_1CD_1, F_1QC, RCE_1, CF_1E_1, RF_1C, and RQF_1.

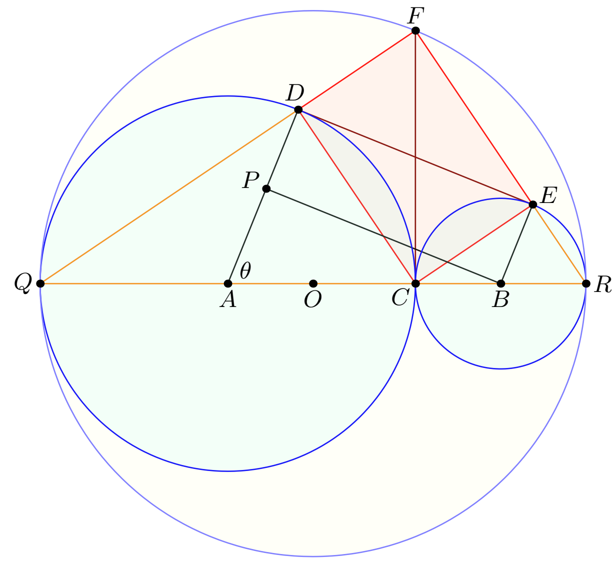


Hence $CD_1F_1E_1$ is a rectangle and

$$\mathrm{m}\angle CQD_1 = \mathrm{m}\angle F_1QC = \mathrm{m}\angle RF_1C = \mathrm{m}\angle CF_1E_1 = \mathrm{m}\angle CD_1E_1.$$

Thus $\angle CQD_1$ and $\angle CD_1E_1$ both intercept minor arc CD_1 in circle Ω_1, implying that line D_1E_1 is tangent to Ω_1 at D_1. Likewise, line D_1E_1 is tangent to Ω_2 at E_1, hence $\ell = \overleftrightarrow{D_1E_1}$. Consider the semicircles with diameters $\overline{QC}$ and $\overline{CR}$ that lie on the same side of $\overleftrightarrow{QR}$ as point F_1. Because these semicircles have a unique common exterior tangent line, it follows that $D = D_1$, $E = E_1$ and $F = F_1$. Then $CQ = 22$, $CR = 10$, and $CF = \sqrt{CQ \cdot CR} = 2\sqrt{55}$. Finally, $QF = \sqrt{QC^2 + FC^2} = 8\sqrt{11}$, and $\sin\angle DEF = \sin\angle CQF = \boldsymbol{\dfrac{\sqrt{5}}{4}}$.

Alternate Solution: Denote by A, B, and O the centers of Ω_1, Ω_2, and Γ, respectively. Let C be the tangency point of Ω_1 and Ω_2, and construct diameters $\overline{QC}$ and $\overline{CR}$ of Ω_1 and Ω_2, as shown.



Observe there is a positive homothety taking Ω_1 to Ω_2 mapping D to E, centered along the intersection of ℓ and line AB. Thus $\triangle QDC \sim \triangle CER$ and moreover, the corresponding sides are parallel. In particular, because $\mathrm{m}\angle QDC = \mathrm{m}\angle CER = 90^\circ$, the lines QD and ER are perpendicular. Because $\overline{QR}$ is a diameter of Γ, it follows that $\overleftrightarrow{QD}$ and $\overleftrightarrow{ER}$ meet at a right angle on Γ at some point G with $\mathrm{m}\angle DGE = 90^\circ$. Because $\mathrm{m}\angle DGE = 90^\circ$, it follows that G lies on the circle with diameter $\overline{DE}$. This circle intersects Γ twice, and the reader can confirm that G is the intersection point closer to D, and thus $G = F$. Recalling also that $\overline{DC} \perp \overline{CE}$ by the same pair of similar triangles, it follows that $CDFE$ is a rectangle. Next, notice that

$$\mathrm{m}\angle DEF = \mathrm{m}\angle CDE = \mathrm{m}\angle CQD = \frac{1}{2}\mathrm{m}\angle DAC.$$

Finally, construct rectangle $DEBP$ so that P lies on $\overline{AD}$. Then $\cos\angle DAC = \cos\angle PAB = \frac{PA}{AB}$, and this yields

$$\begin{aligned} \sin\angle DEF &= \sin\frac{1}{2}\angle DAC = \sin\frac{1}{2}\angle PAB \\ &= \sqrt{\frac{1-\cos\angle PAB}{2}} = \sqrt{\frac{1-\frac{PA}{AB}}{2}} = \sqrt{\frac{1-\frac{11-5}{11+5}}{2}} = \frac{\sqrt{5}}{4}. \end{aligned}$$

Problem 9. Nine distinct circles are drawn in the plane so that at most half of the pairs of these circles intersect. The circles divide the plane into N regions of finite area. Compute the maximum possible value of N.

Solution 9. First, notice that if any circles are concurrent or tangent, then one may slightly increase the radius of any such circle while increasing the number of regions formed, without violating any conditions. Thus assume henceforth that no three circles are concurrent and no two circles are tangent. Let $k \leq \frac{1}{2}\binom{9}{2} = 18$ denote the number of pairs of circles that intersect in two points. The idea is to use the following version of Euler's Formula on the graph G whose vertices are intersection points and whose edges are arcs, and c is the number of connected components of G:

$$V - E + F = 2 + (c - 1).$$

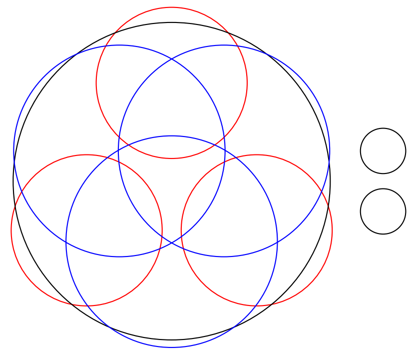
For convenience, if there are any circles with no intersections at all, then consider this as a connected component with 0 vertices and 0 edges (in which case the formula still holds). Notice that $V = 2k$, because every pair of intersecting circles yields two vertices. Moreover, because every vertex in G has degree 4, it follows that $E = 2V = 4k$. Thus

$$F = E - V + 2 + (c - 1) = 2k + 2 + (c - 1).$$

Finally, only one region has infinite area (the unbounded face), thus

$$N = F - 1 = 2k + 1 + (c - 1).$$

Now consider the connected components of the graph G, which correspond to groups of circles. If there are at least four groups, then the maximum possible value of k is $\binom{6}{2} + \binom{1}{2} + \binom{1}{2} + \binom{1}{2} = 15$. So for $16 \leq k \leq 18$, it follows that $c \leq 3$, and thus $N \leq 39$. For $k \leq 15$, it is enough to use $c \leq 9$ to obtain $N \leq 39$ as well. The following example shows that $N = 39$ indeed works because it satisfies $c = 3$ and $k = 18$. (Note that $k = \binom{7}{2} - 3 = 18$ because the three red circles do not intersect.) Hence the final answer is $N = \mathbf{39}$.



Problem 10. For a positive integer k, let $S(k)$ denote the sum of the digits of k. Compute the number of seven-digit positive integers n such that $S(n) + 4S(2n) = S(36n)$.

Solution 10. The given equation can be rewritten as

$$S(10n) + S(20n) + 3S(2n) = S(36n).$$

The solution relies on the following lemma: for any integers a and b,

$$S(a + b) \leq S(a) + S(b),$$

with equality if and only if there are no carries in the addition of a and b. This follows from the general fact that $S(a + b) = S(a) + S(b) - 9c$, where c is the number of carries in the addition of a and b. Notice that

$$10n + 20n + 2n + 2n + 2n = 36n.$$

Using the lemma, the problem is equivalent to finding the number of integers n such that the five-term addition $10n + 20n + 2n + 2n + 2n$ has no carries. Let $2n = \underline{d_1}\,\underline{d_2} \ldots \underline{d_k}$ and let $10n = \underline{e_0}\,\underline{e_1} \ldots \underline{e_{k-1}}\,\underline{0}$, where the d_i's and e_i's are digits. Note that d_1 could possibly be zero. The addition table for the desired sum is shown below.

$$\begin{array}{r|ccccccc}
2n & & d_1 & d_2 & \ldots & d_{k-2} & d_{k-1} & d_k \\
2n & & d_1 & d_2 & \ldots & d_{k-2} & d_{k-1} & d_k \\
2n & & d_1 & d_2 & \ldots & d_{k-2} & d_{k-1} & d_k \\
10n & e_0 & e_1 & e_2 & \ldots & e_{k-2} & e_{k-1} & 0 \\
20n & d_1 & d_2 & d_3 & \ldots & d_{k-1} & d_k & 0
\end{array}$$

It immediately follows that $d_i \in \{0, 1, 2\}$ for each i and this is assumed henceforth. Then for $i = 0, 1, \ldots, k-1$, note that

$$e_i = \begin{cases} 0 & \text{if } d_i \neq 1 \text{ and } d_{i+1} \neq 2 \\ 1 & \text{if } d_i \neq 1 \text{ and } d_{i+1} = 2 \\ 5 & \text{if } d_i = 1 \text{ and } d_{i+1} \neq 2 \\ 6 & \text{if } d_i = 1 \text{ and } d_{i+1} = 2. \end{cases}$$

It follows that a choice of $d_i \in \{0, 1, 2\}$ for each i is valid if and only if whenever $d_i = 1$, $d_{i+1} \neq 2$. In other words, the set of valid choices of $2n$ consists of even integers with digits only 0, 1, 2, such that the string 12 never appears. Letting F_i denote the ith term of the Fibonacci sequence, the following results can be established by induction on $d \geq 1$:

- the number of such d-digit strings starting with 0 (allowing leading zeros) is F_{2d-1};
- the number of such d-digit strings starting with 1 is F_{2d-2}; and
- the number of such d-digit strings starting with 2 is F_{2d-1}.

Finally, if n is an r-digit number, then $2n$ must be either an r-digit number beginning with 2, or an $(r+1)$-digit number beginning with 1. There are $F_{2r-1} + F_{2r} = F_{2r+1}$ such numbers that satisfy the desired constraints, and with $r = 7$, the answer is $F_{15} = \mathbf{610}$.