CT ARML Team, 2025
Team Selection Test 2

1. Let $N = \underline{9}\,\underline{9}\,\underline{9}\,\underline{9} \ldots \underline{9}\,\underline{9}\,\underline{9}$ be the number with 100 digits, all of which are 9s. Compute the <u>sum</u> of the digits of the number N^2.

2. How many ordered triples (a, b, c) of odd positive integers satisfy the equation $a + b + c = 25$?

3. In triangle ABC, $BC = a$, $CA = b$, and $AB = c$. If a, b, c are the roots of the equation $x^3 - 13x^2 + 52x - 61 = 0$, then the value of the quantity

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$$

is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

4. Let XYZ be a right triangle with right angle Y. Point W lies on side $\overline{YZ}$ and line $\overline{XW}$ bisects angle X. Given that XY, YW, and WZ are all integers, compute the minimum possible value of $XY + YW + WZ$.

5. Suppose that the polynomial function P can be represented in the form $P(x) = x^4 + ax^3 + bx^2 + cx + d$, where a, b, c, d are constants, and that $P(1) = 10$, $P(2) = 20$, and $P(3) = 30$. Compute $P(5) + P(-1)$.

6. Each of the six faces of a cube is painted with one of four colors, so that no adjacent faces have the same color. Two colored cubes are considered to be the same if they can be oriented so that faces with the same directional orientation have the same colors. How many distinct colored cubes are there?

7. Points A, B, C and D lie on a circle, in that order. $DC = 4$, $DB = 10$, $DA = 11$, and $m\angle CDB = m\angle BDA$. Let the radius of the circle be r. Compute r^2.

8. The Fibonacci sequence $F_1, F_2, F_3, \ldots$ is defined by $F_1 = F_2 = 1$ and, for $n \geq 3$, $F_n = F_{n-1} + F_{n-2}$. The value of the sum

$$\sum_{n=1}^{\infty} \left[F_n \left(\frac{1}{2^n} + \frac{1}{3^n} \right) \right]$$

is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

9. Suppose that a and b are real numbers such that $a^3b - ab^3 = 20$ and $(a^2 - b^2)^2 - 4a^2b^2 = 18$. Compute $(a^2 + b^2)^2$.

10. Let $ABCD$ be a convex quadrilateral with $DA = DB = DC$ and $\overline{AD} \parallel \overline{BC}$. The perpendicular bisector of line segment $\overline{CD}$ meets line $\overline{AB}$ at E such that B is between A and E. If the measure of $\angle BCE$ is twice the measure of $\angle CED$, then the measure of $\angle BCE$ in degrees is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

CT ARML Team, 2025
Team Selection Test 2
Answers

1. 900
2. 78
3. 187
4. 14
5. 184
6. 10
7. 32
8. 18
9. 82
10. 367

CT ARML Team, 2025
Team Selection Test 2
Solutions

1. Let $N = \underline{9}\,\underline{9}\,\underline{9}\,\underline{9} \ldots \underline{9}\,\underline{9}\,\underline{9}$ be the number with 100 digits, all of which are 9s. Compute the <u>sum</u> of the digits of the number N^2.

<u>Solution</u>:

$$\begin{aligned} N &= \underline{9999}\ldots\underline{9999} \\ &= 9\cdot(1+10+10^2+\cdots+10^{99}) \\ &= 9\cdot\frac{10^{100}-1}{10-1} \\ &= 10^{100}-1 \end{aligned}$$

Therefore,

$$\begin{aligned} N^2 &= 10^{200} - 2\times 10^{100} + 1 \\ &= (10^{100}-2)\times 10^{100}+1 \\ &= (\quad \underbrace{\underline{999}\ldots\underline{8}}_{\text{99 9s followed by an 8}} \quad)\times 10^{100}+1 \end{aligned}$$

So the sum of digits of N^2 is $9\times 100 - 1 + 1 = 900$.

2. How many ordered triples (a, b, c) of odd positive integers satisfy the equation $a + b + c = 25$?

Solution. Use generating functions. The generating functions for odd positive integers is $x \times (1 + x^2 + x^4 + \cdots) = \frac{x}{1-x^2}$

So we are looking for the coeff of x^{25} in $\frac{x^3}{(1-x^2)^3}$.

This is the same as the coeff of y^{11} in $(1-y)^{-3}$

The answer is $(-1)^{11}\times\binom{-3}{11}$ which is $\frac{13!}{2!11!} = 78$

3. In triangle ABC, $BC = a$, $CA = b$, and $AB = c$. If a, b, c are the roots of the equation $x^3 - 13x^2 + 52x - 61 = 0$, then the value of the quantity

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$$

is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

Solution:
Using the cosine formula, we have:

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc} = \frac{(a + b + c)^2 - 2(ab + bc + ac)}{2abc}$$

Using Vieta's theorem, $a + b + c = 13, ab + bc + ac = 52, abc = 61.$

Therefore, the value is $\frac{13^2-2\cdot 52}{2\cdot 61} = \frac{65}{122}, p + q = 187.$

4. Let XYZ be a right triangle with right angle Y. Point W lies on side $\overline{YZ}$ and line $\overline{XW}$ bisects angle X. Given that XY, YW, and WZ are all integers, compute the minimum possible value of $XY + YW + WZ$.

Solution:
Let $YW = a, WZ = b, XY = x, XZ = y$. By the angle bisector theorem, $\frac{y}{b} = \frac{x}{a}$. By the Pythagorean Theorem, $x^2 + (a + b)^2 = y^2$. Eliminating y, we get:

$$x = a \cdot \sqrt{\frac{b + a}{b - a}}.$$

For a, b, x to all be integers, we must have $\frac{b+a}{b-a} = n^2$ where n is an integer. Let $k = b - a$, then we have $a = \frac{k(n^2-1)}{2}, b = \frac{k(n^2+1)}{2}$. The smallest n is 2 so that $a > 0$, the smallest k is 2 so that a and b are integers. This gives $a = 3, b = 5, x = 6$.

The answer is therefore $3 + 5 + 6 = 14$.

5. Suppose that the polynomial function P can be represented in the form $P(x) = x^4 + ax^3 + bx^2 + cx + d$, where a, b, c, d are constants, and that $P(1) = 10$, $P(2) = 20$, and $P(3) = 30$. Compute $P(5) + P(-1)$.

Solution:
Using $P(1) = 10, P(2) = 20, P(3) = 30$, we have:

$$\begin{array}{lrrrrl} (1) & a + & b + & c + & d = & 9, \\ (2) & 8a + & 4b + & 2c + & d = & 4, \\ (3) & 27a + & 9b + & 3c + & d = & -51. \end{array}$$

We also have:

$P(5) + P(-1) = 626 + 124a + 26b + 4c + 2d.$

To find y, z, w so that $124a + 26b + 4c + 2d = (1) \cdot y + (2) \cdot z + (3) \cdot w$, we match up coefficients on a, b, c, d:

$$\begin{cases} y + 8z + 27w = 124, \\ y + 4z + \ \ 9w = \ \ 26, \\ y + 2z + \ \ 3w = \ \ \ \ 4, \\ y + \ \ z + \ \ \ \ w = \ \ \ \ 2. \end{cases}$$

Solving this set of equations through elimination, we find that $y = 9, z = -16, w = 9$ are the solutions. Therefore:

$$P(5) + P(-1) = 626 + 9 \cdot 9 - 16 \cdot 4 - 9 \cdot 51 = 184.$$

6. Each of the six faces of a cube is painted with one of four colors, so that no adjacent faces have the same color. Two colored cubes are considered to be the same if they can be oriented so that faces with the same directional orientation have the same colors. How many distinct colored cubes are there?

Solution:
If color #1 is not used, there is exactly one way to color the cube.

Next we can assume that the top face has color #1.

If the bottom face also has color #1, then there are six ways to color the side faces:

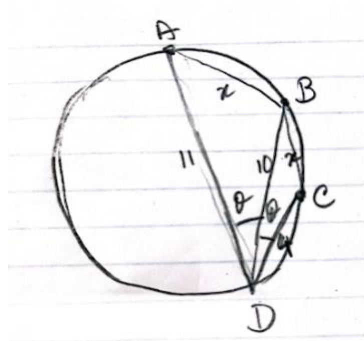
[#2, #3, #2, #3]; [#2, #4, #2, #4]; [#3, #4, #3, #4];
[#2, #3, #2, #4]; [#3, #2, #3, #4]; [#4, #2, #4, #3].

If the bottom face has any of the other three colors, in each case there is exactly one way to color the side faces.

So the total number of possibilities is $1 + 6 + 3 = 10$.

7. Points A, B, C and D lie on a circle, in that order. $DC = 4$, $DB = 10$, $DA = 11$, and m$\angle CDB$ = m$\angle BDA$. Let the radius of the circle be r. Compute r^2.

Solution:



Solution. Using the rule of sines in triangles ABD, BCD

$$2r = \frac{x}{\sin\theta} = \frac{11}{\sin\angle ABD} = \frac{10}{\sin\angle DAB} = \frac{4}{\sin\angle DBC} = \frac{AC}{\sin 2\theta}$$

But since ABCD is cyclic, $AC \times BD = CD \times AB + AD \times BC$, or

$$10AC = 15x$$

Thus

$$\frac{x}{\sin\theta} = \frac{(15/10)x}{\sin 2\theta} = \frac{(3/2)x}{\sin 2\theta}$$

Solving for θ, $\sin\theta = \frac{\sqrt{7}}{4}$ and $\cos\theta = \frac{3}{4}$.

Noticing that $\angle DAB = 180° - \theta - \angle ABD$, and using (derived above) $\frac{11}{\sin\angle ABD} = \frac{10}{\sin\angle DAB}$, we get $\sin\angle ABD = \frac{11}{8\sqrt{2}}$. Therefore

$$\begin{aligned} 2r &= \frac{11}{\sin\angle ABD} \\ &= 8\sqrt{2} \end{aligned}$$

Hence

$$\begin{aligned} r &= 4\sqrt{2} \\ \text{Hence, } r^2 &= 32 \end{aligned}$$

8. The Fibonacci sequence $F_1, F_2, F_3, \ldots$ is defined by $F_1 = F_2 = 1$ and, for $n \geq 3$, $F_n = F_{n-1} + F_{n-2}$. The value of the sum

$$\sum_{n=1}^{\infty} \left[F_n \left(\frac{1}{2^n} + \frac{1}{3^n} \right) \right]$$

is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

<u>Solution</u>:

Let $a = \sum_{n=1}^{\infty} \frac{F_n}{2^n}$, and $b = \sum_{n=1}^{\infty} \frac{F_n}{3^n}$.

$$a = \frac{F_1}{2} + \frac{F_2}{2^2} + \frac{F_3}{2^3} + \frac{F_4}{2^4} + \cdots$$

$$2a = F_1 + \frac{F_2}{2} + \frac{F_3}{2^2} + \frac{F_4}{2^3} + \frac{F_5}{2^4} + \cdots$$

$$2a - a = F_1 + \frac{F_2 - F_1}{2} + \frac{F_3 - F_2}{2^2} + \frac{F_4 - F_3}{2^3} + \frac{F_5 - F_4}{2^4} + \cdots$$

$$= 1 + 0 + \frac{F_1}{2^2} + \frac{F_2}{2^3} + \frac{F_3}{2^4} + \cdots = 1 + \frac{a}{2}$$

So $a = 2$.

Similarly, $3b - b = 1 + \frac{b}{3}$ and we get $b = \frac{3}{5}$.

$a + b = \frac{13}{5}$, and $p + q = 18$. The answer is $\boxed{18}$.

9. Suppose that a and b are real numbers such that $a^3b - ab^3 = 20$ and $(a^2 - b^2)^2 - 4a^2b^2 = 18$. Compute $(a^2 + b^2)^2$.

<u>Solution</u>:

Simplify the second equation we get $a^4 - 6a^2b^2 + b^4 = 18$. If we multiple the first equation by 4, we get $4a^3b - 4ab^3 = 80$. The polynomials at the left hand side of these two equations look familiar that is the expansion of $(a + b)^4$ but with negative terms. If we add an *i* in front of *b*, it will match. Indeed,

$$(a + bi)^4 = a^4 - 6a^2b^2 + b^4 + (4a^3b - 4ab^3)i = 18 + 80i$$

Thus,

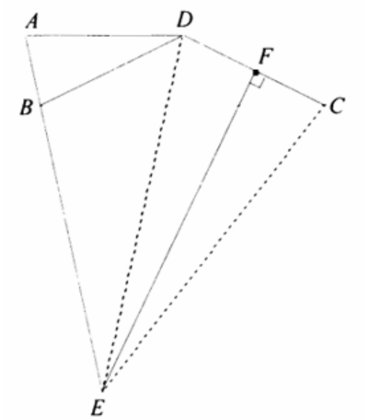
$$a^2 + b^2 = |a + bi|^2 = \sqrt[4]{18^2 + 80^2} = \sqrt{82}$$

The answer is $\boxed{82}$.

10. Let $ABCD$ be a convex quadrilateral with $DA = DB = DC$ and $\overline{AD} \parallel \overline{BC}$. The perpendicular bisector of line segment $\overline{CD}$ meets line $\overline{AB}$ at E such that B is between A and E. If the measure of $\angle BCE$ is twice the measure of $\angle CED$, then the measure of $\angle BCE$ in degrees is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

Solution:

DA = DB so ΔADB is an isosceles triangle. Let $m\angle ADB = 2x$, $then\ m\angle DAB = m\angle DBA = 90° - x.$



$\overline{AD} \parallel \overline{BC}$, and DB = DC, so $m\angle DBC = m\angle DCB = m\angle ADB = 2x.$

Next we prove CBDE is co-cyclic.

Let circle O denote the circumcircle of ΔBCD and assume that O intersects $\overline{AB}$ at two distinct points, B and E'. Since B, C, D and E' lie on circle O, we can say that

$m\angle CE'D = m\angle CBD = 2x$ and $m\angle DCE' = m\angle DBA = 90° - x.$

Thus $m\angle CDE' = 180° - m\angle CE'D - m\angle DCE' = 180° - 2x - (90° - x) = 90° - x.$

So $m\angle CDE' = m\angle DCE' = 90° - x.$

This means that $\Delta CDE'$ is an isosceles triangle and E' is on the perpendicular bisector of $\overline{CD}$. Plus E' is on line $\overline{AB}$, we can conclude that E = E'. This proves CBDE is cocyclic.

From the given condition, $m\angle BCE = 2\, m\angle CED = 4x$, and when we also have

$$m\angle BCE = m\angle DCE - m\angle DCB = 90° - x - 2x = 90° - 3x$$

So $4x = 90° - 3x$ and $x = \frac{90}{7}$. Thus $m\angle BCE = 4x = \frac{360}{7}$.

The answer is $\boxed{367}$.

4 Power Question 2021: Complex Triangles

Instructions: The power question is worth 50 points; each part's point value is given in brackets next to the part. To receive full credit, the presentation must be legible, orderly, clear, and concise. If a problem says "list" or "compute," you need not justify your answer. If a problem says "determine," "find," or "show," then you must show your work or explain your reasoning to receive full credit, although such explanations do not have to be lengthy. If a problem says "justify" or "prove," then you must prove your answer rigorously. Even if not proved, earlier numbered items may be used in solutions to later numbered items, but not vice versa. Pages submitted for credit should be NUMBERED IN CONSECUTIVE ORDER AT THE TOP OF EACH PAGE in what your team considers to be proper sequential order. PLEASE WRITE ON ONLY ONE SIDE OF THE ANSWER PAPERS. Put the TEAM NUMBER (not the team name) on the cover sheet used as the first page of the papers submitted. Do not identify the team in any other way.

A *complex triangle* (hereafter called a *c-triangle*) is an unordered list of three complex numbers, $[a, b, c]$, with $abc \neq 0$. The values a, b, and c are the *side lengths* of the c-triangle. Two c-triangles are considered the same (or *congruent*) if and only if their side lengths are the same, regardless of order. For example, $[1, 1, 3]$ and $[1, 3, 1]$ represent congruent c-triangles. A *complex angle* (hereafter called a *c-angle*) is an ordered triple (a, b, c) whose elements are complex numbers, again with $abc \neq 0$. The *c-cosine* of a c-angle is defined as

$$\operatorname{ccos}(a, b, c) = \frac{b^2 + c^2 - a^2}{2bc}.$$

Two c-angles are *equal* if and only if they have the same c-cosine. In particular, note that the c-angles (a, b, c) and (a, c, b) are equal.

Every c-triangle $[a, b, c]$ has three c-angles, which are denoted by $\angle A = (a, b, c)$, $\angle B = (b, a, c)$, and $\angle C = (c, a, b)$. A c-angle $\angle P$ is called *proper* if $|\operatorname{ccos}(\angle P)| \leq 1$, and *improper* otherwise. (Note: The absolute value of a complex number $x + yi$, where x and y are real, is $\sqrt{x^2 + y^2}$.)

The *squarea* ("squared area") of a c-triangle is defined by the formula

$$\operatorname{sqar}[a, b, c] = s(s - a)(s - b)(s - c), \quad \text{where } s = \frac{a + b + c}{2}.$$

1. **a.** For the c-triangle $[2, 3, 4]$, compute the c-cosines of its three c-angles. [3 pts]

b. Suppose a c-triangle $[i, 1, c]$ exists with $\operatorname{ccos}(i, 1, c) = 1$ and $\operatorname{ccos}(1, i, c) = 1$. Compute c. [2 pts]

2. For each of the following properties, list an example of a c-triangle $[a, b, c]$ satisfying that property.

a. $\operatorname{ccos}(\angle A) = 0$ [1 pt]

b. $\operatorname{ccos}(\angle A) = 2$ [1 pt]

c. $\operatorname{ccos}(\angle A) = i$ [1 pt]

d. $\operatorname{sqar}[a, b, c] = 36$ [1 pt]

e. $\operatorname{sqar}[a, b, c] = -36$ [1 pt]

f. $\operatorname{sqar}[a, b, c] = -\frac{3}{4}$ [1 pt]

3. Find the set of all complex numbers z for which $(1, 1, z)$ is a proper c-angle. [2 pts]

4. List two noncongruent c-triangles $[a_1, b_1, c_1]$ and $[a_2, b_2, c_2]$ such that $a_1 = a_2$, $b_1 = b_2$, and $\angle C_1 = \angle C_2$. (Doing this will show that "SAS congruence" does not hold for c-triangles.) [3 pts]

5. Find an example of each of the following.

a. A c-triangle that has two improper c-angles and one proper c-angle. [2 pts]

b. A c-triangle that has two proper c-angles and one improper c-angle. [2 pts]

6. Show that if every c-angle of the c-triangle $[a, b, c]$ is proper, then $|b| + |c| \geq |a|$. [3 pts]

7. **a.** Show that $\operatorname{sqar}[a, b, c] = \frac{1}{4}b^2c^2(1 - (\operatorname{ccos}(\angle A))^2)$. [4 pts]

   b. Show that for any c-triangle $[a, b, c]$,

$$\frac{1 - (\operatorname{ccos}(\angle A))^2}{a^2} = \frac{1 - (\operatorname{ccos}(\angle B))^2}{b^2}.$$ [2 pts]

8. Show that if two c-triangles $[a_1, b_1, c_1]$ and $[a_2, b_2, c_2]$ have corresponding c-angles equal (that is, $\angle A_1 = \angle A_2$, $\angle B_1 = \angle B_2$, and $\angle C_1 = \angle C_2$), and none of those c-angles has c-cosine equal to ± 1, then $\dfrac{a_2}{a_1} = \dfrac{b_2}{b_1}$. (That is, one could call these c-triangles "similar".) [4 pts]

A *c-configuration* consists of a finite set S, whose elements will be called *c-points* and denoted by capital letters. In addition, for each pair of distinct c-points in a c-configuration, a nonzero complex number called the *distance* is assigned to the pair. The distance between the unordered c-points $A, B \in S$ will be denoted by either $d(A, B)$ or $d(B, A)$, interchangeably.

If A, B, and C are c-points, then the c-triangle $[d(A, B), d(A, C), d(B, C)]$ will be denoted by $\triangle ABC$, and the c-angle $\big(d(A, C), d(A, B), d(B, C)\big)$ by $\angle ABC$. Note that A, B, and C denote c-points, <u>not</u> c-angles.

9. Let A, B, and C be c-points in a c-configuration. Then the ordered triple (A, B, C) is called *collinear* if $d(A, B) + d(B, C) = d(A, C)$. Show that if (A, B, C) is collinear, then $\operatorname{ccos}(\angle ABC) = -1$. [3 pts]

10. Suppose that A, B, C, and D are c-points in a c-configuration, and (B, D, C) is collinear. Prove that $\operatorname{ccos}(\angle ABD) = \operatorname{ccos}(\angle ABC)$ if and only if $\operatorname{ccos}(\angle ACD) = \operatorname{ccos}(\angle ACB)$. [6 pts]

If A, B, C, D are c-points in a c-configuration, then define the ordered pair (A, D) to be a *c-cevian* of $\triangle ABC$ if (B, D, C) is collinear and the two equations in Problem 10 are satisfied.

11. Given that (A, D) is a c-cevian of $\triangle ABC$, prove that $\operatorname{ccos}(\angle ADC) = -\operatorname{ccos}(\angle ADB)$. [4 pts]

12. Given that (A, D) is a c-cevian of $\triangle ABC$ and $\operatorname{ccos}(\angle ADC) = 0$, prove that

$$\operatorname{sqar} \triangle ABC = \frac{1}{4}\big(d(A, D)\big)^2 \cdot \big(d(B, C)\big)^2.$$ [4 pts]

5 Solutions to Power Question

1. **a.** The three c-cosines are $\operatorname{ccos}(2,3,4) = \frac{3^2+4^2-2^2}{2\cdot 3\cdot 4} = \frac{7}{8}$, $\operatorname{ccos}(3,2,4) = \frac{2^2+4^2-3^2}{2\cdot 2\cdot 4} = \frac{11}{16}$, and $\operatorname{ccos}(4,3,2) = \frac{3^2+2^2-4^2}{2\cdot 3\cdot 2} = -\frac{1}{4}$.

b. The answer is $c = 1+i$. By the definition of c-cosine, $\operatorname{ccos}(i,1,c) = \frac{1^2+c^2-i^2}{2c} = 1$ and $\operatorname{ccos}(1,i,c) = \frac{i^2+c^2-1^2}{2ic} = 1$. This implies $1^2+c^2-i^2 = 2c$ and $i^2+c^2-1^2 = 2ic$. Add to obtain $2c^2 = 2c(1+i)$. Because $c \neq 0$, divide both sides by $2c$ to obtain $c = 1+i$. It can be verified that $c = 1+i$ does indeed satisfy the conditions of the problem.

2. There are many possible c-triangles for each part. At least one example is given below for each.

a. $[5,3,4]$, or in general, any c-triangle in which $a^2 = b^2+c^2$

b. $[\sqrt{2}i,1,1]$, or in general, any c-triangle in which $b^2+c^2-4bc = a^2$

c. $[\sqrt{2},i,1]$, or in general, any c-triangle in which $b^2+c^2-2ibc = a^2$

d. $[3,4,5]$

e. $[3\alpha,4\alpha,5\alpha]$, where $\alpha = \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$. (Note that $\alpha^4 = -1$.)

f. $[1+i,1+i,1+i]$

3. By definition, $\operatorname{ccos}(1,1,z) = \frac{1^2+z^2-1^2}{2\cdot 1\cdot z} = \frac{1}{2}z$. Therefore S is the set of all z with $|z| \leq 2$; in other words, the closed disk of radius 2 centered at 0 in the complex plane.

4. One such pair of c-triangles is $[1,1,3]$ and $[1,1,-3]$. In fact, because the formula for c-cosine is an even function in the first argument, it follows that $\operatorname{ccos}(c,a,b) = \operatorname{ccos}(-c,a,b)$. Therefore any two c-triangles of the form $[a,b,c]$ and $[a,b,-c]$ will satisfy the given condition. In fact, by setting $\operatorname{ccos}(c_1,a,b) = \operatorname{ccos}(c_2,a,b)$, it can be shown that these are the *only* examples.

5. **a.** One such c-triangle is $[1,i,2]$. The c-cosines of its three c-angles are $\frac{1+(-1)-4}{2i} = 2i$, $\frac{1+4-(-1)}{4} = \frac{3}{2}$, and $\frac{4+(-1)-1}{4i} = -\frac{1}{2}i$. The first two c-angles are improper, and the third is proper.

b. One such c-triangle is $[1,1,i]$. The c-cosines of its three c-angles are $\frac{1+(-1)-1}{2i} = \frac{1}{2}i$ (twice) and $\frac{1+1-(-1)}{2} = \frac{3}{2}$. The first two c-angles are proper, and the third is improper.

6. First, note that for any complex numbers x and y, the standard triangle inequality holds (that is, $|x+y| \leq |x|+|y|$). Applying the triangle inequality to y and $x-y$ gives $|x| \leq |y|+|x-y|$, so $|x-y| \geq |x|-|y|$. (This will be referred to as the *subtractive triangle inequality.*)

Now consider a c-triangle $[a,b,c]$, all of whose angles are proper. By definition, the c-angle (a,b,c) is proper, so $|b^2+c^2-a^2| \leq |2bc|$. By reversing the sign in the first absolute value expression, $|a^2-b^2-c^2| \leq |2bc|$.

Using the subtractive triangle inequality from above twice, $|a^2-b^2-c^2| \geq |a^2-b^2|-|c^2| \geq |a^2|-|b^2|-|c^2|$. Therefore

$$|a|^2-|b|^2-|c|^2 \leq 2\cdot|b|\cdot|c|.$$

Rearranging and factoring gives $|a|^2 \leq (|b|+|c|)^2$. Then because both sides are positive real numbers, taking the square root yields $|a| \leq |b|+|c|$, as desired.

7. **a.** Observe that $16\operatorname{sqar}[a,b,c] = (a+b+c)(-a+b+c)(a-b+c)(a+b-c)$. Expand the right-hand side by first multiplying out the first two and last two factors to obtain:

$$16\operatorname{sqar}[a,b,c] = (-a^2+b^2+c^2+2bc)(a^2-b^2-c^2+2bc) = 4b^2c^2-(b^2+c^2-a^2)^2.$$

Now factor out $4b^2c^2$ from the right-hand side, yielding:

$$16\operatorname{sqar}[a,b,c] = 4b^2c^2\left(1-\frac{(b^2+c^2-a^2)^2}{(2bc)^2}\right) = 4b^2c^2(1-(\operatorname{ccos}(\angle A))^2).$$

Dividing both sides by 16 yields the desired equality.

b. From the result of part (a),

$$4\,\mathrm{sqar}[a,b,c] = b^2c^2(1-(\mathrm{ccos}(\angle A))^2) = a^2c^2(1-(\mathrm{ccos}(\angle B))^2).$$

Dividing both sides by $a^2b^2c^2$ yields the desired equality.

8. Rearranging the result of Problem 7(b) and relying on the assumption that $\mathrm{ccos}(\angle B_1) \neq \pm 1$ yields:

$$\frac{1-(\mathrm{ccos}(\angle A_1))^2}{1-(\mathrm{ccos}(\angle B_1))^2} = \frac{a_1^2}{b_1^2}.$$

However, because $\mathrm{ccos}(\angle A_1) = \mathrm{ccos}(\angle A_2)$ and $\mathrm{ccos}(\angle B_1) = \mathrm{ccos}(\angle B_2)$, it follows that

$$\frac{a_1^2}{b_1^2} = \frac{a_2^2}{b_2^2}, \qquad \text{so} \qquad \frac{a_2}{a_1} = \pm\frac{b_2}{b_1}.$$

Similar arguments show that $\frac{a_2}{a_1} = \pm\frac{c_2}{c_1}$, so for some $k \neq 0$, $a_2 = ka_1$, $b_2 = ekb_1$, and $c_2 = fkc_1$, where $e = \pm 1$ and $f = \pm 1$.

It remains to show that $e = 1$, for then $\frac{a_2}{a_1} = \frac{b_2}{b_1} = k$. Because $\mathrm{ccos}(\angle C_1) = \mathrm{ccos}(\angle C_2)$ and $e^2 = f^2 = 1$, it follows that

$$\frac{a_1^2+b_1^2-c_1^2}{2a_1b_1} = \frac{(ka_1)^2+(ekb_1)^2-(fkc_1)^2}{2(ka_1)(ekb_1)} = \frac{a_1^2+b_1^2-c_1^2}{2ea_1b_1}.$$

As long as $a_1^2+b_1^2 \neq c_1^2$, it follows immediately that $e = 1$.

In the remaining case where $a_1^2+b_1^2 = c_1^2$, note that $a_1^2+c_1^2 \neq b_1^2$ and $b_1^2+c_1^2 \neq a_1^2$, so the argument above suffices to show that

$$\frac{a_2}{a_1} = \frac{c_2}{c_1} \qquad \text{and} \qquad \frac{b_2}{b_1} = \frac{c_2}{c_1},$$

from which the desired conclusion follows.

9. For convenience, let $x = d(A,B)$ and $y = d(B,C)$, so $d(A,C) = x+y$. Then by definition,

$$\mathrm{ccos}(\angle ABC) = \frac{x^2+y^2-(x+y)^2}{2xy} = \frac{-2xy}{2xy} = -1.$$

10. For convenience, let $a = d(B,C)$, $b = d(A,C)$, $c = d(A,B)$, $x = d(A,D)$, $y = d(B,D)$, and $z = d(C,D)$.

Claim: Both given equations hold if and only if the following equation holds:

$$c^2z + b^2y = ayz + x^2a. \tag{$*$}$$

Proof: The following argument shows that the above equation is equivalent to $\mathrm{ccos}(\angle ABD) = \mathrm{ccos}(\angle ABC)$; the other equivalence follows by exchanging all instances of B and C, b and c, and y and z in this argument. Because performing the exchange of b with c and y with z results in $(*)$, it follows that $\mathrm{ccos}(\angle ABD) = \mathrm{ccos}(\angle ABC)$ if and only if $\mathrm{ccos}(\angle ACD) = \mathrm{ccos}(\angle ACB)$, as desired.

First, apply the definition of the c-cosine of a c-angle, to obtain

$$\mathrm{ccos}(\angle ABD) = \mathrm{ccos}(\angle ABC) \iff \frac{c^2+y^2-x^2}{2cy} = \frac{a^2+c^2-b^2}{2ac}.$$

Now, multiply by $2acy$, which yields

$$\mathrm{ccos}(\angle ABD) = \mathrm{ccos}(\angle ABC) \iff c^2a + y^2a - x^2a = a^2y + c^2y - b^2y.$$

Rearrange the terms in the second equation and factor, to obtain

$$\mathrm{ccos}(\angle ABD) = \mathrm{ccos}(\angle ABC) \iff c^2(a-y) + b^2y = ay(a-y) + x^2a.$$

Finally, because (B, D, C) is collinear, it follows from the definition that $a = y + z$, and substituting $a - y = z$ gives

$$\operatorname{ccos}(\angle ABD) = \operatorname{ccos}(\angle ABC) \iff c^2z + b^2y = ayz + x^2a,$$

as desired. Note that all of these algebraic steps are reversible, so the equivalence is preserved at each stage.

11. Define a new c-configuration on four c-points $\{A', B', C', D'\}$, having all of its distances the same as those of the corresponding pairs of c-points in the original c-configuration with $\{A, B, C, D\}$ (that is, $d(A, B) = d(A', B')$, and so forth), except that $d(C', D') = -d(C, D)$.

Now note that $\operatorname{ccos}(\angle A'B'C') = \operatorname{ccos}(\angle ABC)$, because $\triangle ABC$ and $\triangle A'B'C'$ are congruent c-triangles. Additionally, $\triangle A'B'D'$ and $\triangle ABD$ are congruent, so $\operatorname{ccos}(\angle A'B'D') = \operatorname{ccos}(\angle ABD)$. Because (A, D) is a c-cevian of $\triangle ABC$, it follows that $\operatorname{ccos}(\angle ABC) = \operatorname{ccos}(\angle ABD)$, and therefore $\operatorname{ccos}(\angle A'B'C') = \operatorname{ccos}(\angle A'B'D')$.

Furthermore, (B', C', D') is collinear because $d(B', D') = d(B, D) = d(B, C) - d(C, D) = d(B', C') + d(C', D')$. Together with the above, it follows that (A', C') is a c-cevian of $\triangle A'B'D'$. Therefore $\operatorname{ccos}(\angle A'D'C') = \operatorname{ccos}(\angle A'D'B')$.

Borrowing the notation from the solution to Problem 10,

$$\operatorname{ccos}(\angle A'D'B') = \operatorname{ccos}(\angle A'D'C') = \frac{(-z)^2 + x^2 - b^2}{2x(-z)} = -\operatorname{ccos}(\angle ADC).$$

Again, from the congruence of $\triangle ABD$ and $\triangle A'B'D'$, it follows that $\operatorname{ccos}(\angle A'D'B') = \operatorname{ccos}(\angle ADB)$. Therefore $\operatorname{ccos}(\angle ADB) = -\operatorname{ccos}(\angle ADC)$, as desired.

12. Using the result of Problem 7(a) and the assumption that (A, D) is a c-cevian,

$$\operatorname{sqar} \triangle ABC = \frac{1}{4} d(A, C)^2 \cdot d(B, C)^2 (1 - (\operatorname{ccos}(\angle ACB))^2) = \frac{1}{4} d(A, C)^2 \cdot d(B, C)^2 (1 - (\operatorname{ccos}(\angle ACD))^2).$$

From Problem 7(b) and the assumption that $\operatorname{ccos}(\angle ADC) = 0$, it follows that

$$1 - (\operatorname{ccos}(\angle ACD))^2 = (1 - \operatorname{ccos}(\angle ADC)) \cdot \frac{d(A, D)^2}{d(A, C)^2} = \frac{d(A, D)^2}{d(A, C)^2}.$$

Therefore

$$\operatorname{sqar} \triangle ABC = \frac{1}{4} d(A, C)^2 \cdot d(B, C)^2 \cdot \frac{d(A, D)^2}{d(A, C)^2} = \frac{1}{4} d(A, D)^2 \cdot d(B, C)^2,$$

as desired.