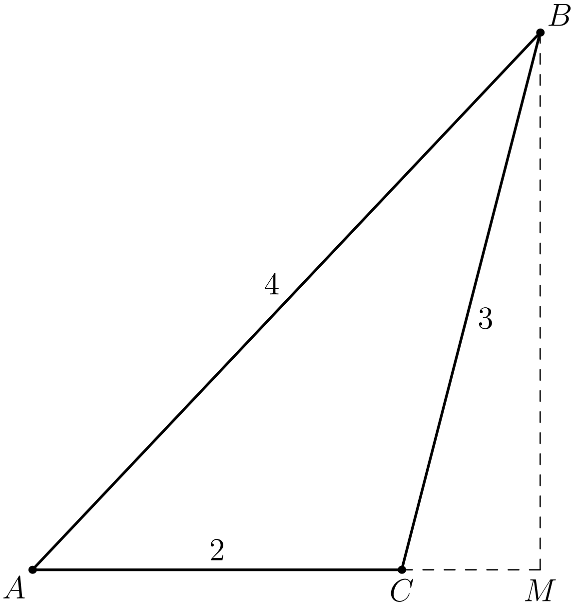
ARML Qualification Test, 2026: Solutions

Problems 15–25

Problem 15

Solution (Girish Prasad).



Let M be the foot of the altitude from B to AC. The desired solid is the result of rotating $\triangle ABM$ around AM and removing the cone that results from rotating $\triangle CBM$ around CM.

We seek the lengths of BM and CM. From the Pythagorean Theorem, we obtain the following equations

$$BM^2 + CM^2 = 9$$

$$BM^2 + (CM+2)^2 = 16$$

$$(BM^2 + (CM+2)^2) - (BM^2 + CM^2) = 16 - 9$$

$$4CM + 4 = 7$$

$$4CM = 3$$

$$CM = \frac{3}{4}$$

$$BM = \sqrt{9 - CM^2} = \frac{3\sqrt{15}}{4}$$

It follows that the volume of the solid is

$$\frac{1}{3}\pi(BM^2)(AM) - \frac{1}{3}\pi(BM^2)(CM)$$

$$= \frac{1}{3}\pi(BM^2)(AM - CM)$$

$$= \frac{1}{3}\pi(BM^2)(2)$$

$$= \frac{1}{3}\pi\left(\frac{135}{16}\right)(2)$$

$$= \frac{45\pi}{8}$$

Hence, the final answer is $45 + 8 = \boxed{53}$.

Problem 16

Solution (Girish Prasad).

Notice that $\frac{(n+1)^2}{n+23}$ is an integer if and only if $(n+1)^2 \equiv 0 \pmod{n+23}$.

Since $n + 1 \equiv -22 \pmod{n+23}$, it follows that

$$(n+1)^2 \equiv (-22)^2 \equiv 484 \pmod{n+23}$$

Hence, $\frac{(n+1)^2}{n+23}$ is an integer if and only if $n + 23$ is a factor of 484. Since we seek positive integers n, it must be that $n + 23 \geq 24$.

The factors of 484 that $n + 23$ may assume are $44, 121, 242,$ and 484. The respective values of n are $21, 98, 219,$ and 461.

The sum of these values is $21 + 98 + 219 + 461 = \boxed{799}$.

Problem 17

Solution (Alex Svoronos).

Notice that $\angle DEP = \angle EFP + \angle EPF = 30^\circ + 25^\circ = 55^\circ$. Now, $\angle PBE = \angle DEP = 55^\circ$. Since $BPER$ is cyclic, we know that $\angle EBR = \angle EPR = 25^\circ$, so $\angle PBR = \angle EBR + \angle EBP = 25^\circ + 55^\circ = 80^\circ$.

Let $\angle BRP = \alpha$. Then, $\angle ABP = \angle BRP = \alpha$, so $\angle BPR = \angle BAP + \angle ABF = \alpha + 35^\circ$. Then, $180^\circ = \angle BPR + \angle BRP + \angle PBR = (\alpha + 35) + \alpha + 80$, so $2\alpha = 65$, and hence $\frac{m}{n} = \frac{65}{2}$, giving us $65 + 2 = \boxed{67}$ as our answer.

Problem 18

Solution (Ayush Rudra).

By Law of Cosines on $\triangle ABC$, we have $AB^2 + AC^2 - 2 \cdot AB \cdot AC \cdot \cos A = BC^2$. This can be rewritten as

$$\begin{aligned}\cos A &= \frac{AB^2 + AC^2 - BC^2}{2 \cdot AB \cdot AC} \\ &= \frac{6^2 + 5^2 - 4^2}{2 \cdot 6 \cdot 5} \\ &= \frac{45}{60} \\ &= \frac{3}{4}.\end{aligned}$$

So, $\sin^2 A = 1 - \cos^2 A = \frac{7}{16}$. Now, we can evaluate what it asks for.

$$\begin{aligned}\sin^6\left(\frac{A}{2}\right) + \cos^6\left(\frac{A}{2}\right) &= \left(\sin^2\left(\frac{A}{2}\right)\right)^3 + \left(\cos^2\left(\frac{A}{2}\right)\right)^3 \\ &= \left(\sin^2\left(\frac{A}{2}\right) + \cos^2\left(\frac{A}{2}\right)\right)\left(\sin^4\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right)\cos^2\left(\frac{A}{2}\right) + \cos^4\left(\frac{A}{2}\right)\right) \\ &= \sin^4\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right)\cos^2\left(\frac{A}{2}\right) + \cos^4\left(\frac{A}{2}\right) \\ &= \left(\sin^2\left(\frac{A}{2}\right) + \cos^2\left(\frac{A}{2}\right)\right)^2 - 3\sin^2\left(\frac{A}{2}\right)\cos^2\left(\frac{A}{2}\right) \\ &= 1 - 3\sin^2\left(\frac{A}{2}\right)\cos^2\left(\frac{A}{2}\right) \\ &= 1 - \frac{3}{4}\left(2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)\right)^2 \\ &= 1 - \frac{3}{4}\sin^2 A \\ &= 1 - \frac{3}{4} \cdot \frac{7}{16} \\ &= \frac{43}{64}.\end{aligned}$$

Thus, $(m, n) = (43, 64)$ and $m + n = \boxed{107}$.

Problem 19

Solution (Alex Svoronos).

Notice that

$$100 = (x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + xz),$$

so $xy + yz + xz = 29$. Now,

$$1764 = (x^2 + y^2 + z^2)^2 = x^4 + y^4 + z^4 + 2(x^2y^2 + x^2z^2 + y^2z^2),$$

hence $x^2y^2 + x^2z^2 + y^2z^2 = 321$. Finally,

$$841 = (xy + yz + xz)^2 = (xz)^2 + (yz)^2 + (xy)^2 + 2x^2yz + 2xy^2z + 2xyz^2 = 321 + 2(xyz)(x + y + z),$$

so $xyz(x + y + z) = 260$, hence $xyz = 26$.

Problem 20

Solution (Jack Whitney-Epstein).

First, factor $3600 = 2^4 3^2 5^2$. Let

$$p = 2^a 3^b 5^c, \quad q = 2^x 3^y 5^z.$$

Since $pq \mid 3600$ and $\gcd(p, q) = 1$, we have the following restrictions on a, b, c, x, y, z:

- $a + x \leq 4, \min(a, x) = 0$
- $b + y \leq 2, \min(b, y) = 0$
- $c + z \leq 2, \min(c, z) = 0$

This has 9 solutions for (a, x): $(0, 0), (0, t)$, and $(t, 0)$ for $1 \leq t \leq 4$. Via similar logic, there are 5 solutions for (b, y) and 5 solutions for (c, z). This gives $9 \cdot 5 \cdot 5 = 225$ values for the pair (p, q). We throw out the useless $p = q = 1$ to get 224 pairs (p, q) with $p \neq q$. Then, for each pair (p, q), exactly one of $\frac{p}{q}$ and $\frac{q}{p}$ is greater than 1. So, half of the remaining cases work. The answer is $\frac{224}{2} = \boxed{112}$.

Problem 21

Solution (Jack Whitney-Epstein).

Note that $\sin(\frac{\pi}{10}) = \cos(\frac{\pi}{2} - \frac{\pi}{10}) = \cos(\frac{2\pi}{5})$. We can derive this value using complex numbers.

Let $t = \cos(\frac{2\pi}{5})$, so that $2t^2 - 1 = \cos(\frac{4\pi}{5})$ by the double-angle formula. Define $\omega = e^{\frac{2\pi i}{5}} = \cos(\frac{2\pi}{5}) + i\sin(\frac{2\pi}{5})$, which is a 5th root of unity. In general, for a complex number z, $z + \bar{z} = 2\text{Re}(z)$. So

$$\cos\left(\frac{2\pi}{5}\right) = \frac{\omega + \bar{\omega}}{2} = \frac{\omega + \omega^4}{2},$$

and

$$\cos\left(\frac{4\pi}{5}\right) = \frac{\omega^2 + \bar{\omega}^2}{2} = \frac{\omega^2 + \omega^3}{2}.$$

Since ω is a primitive 5th root of unity,

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$$

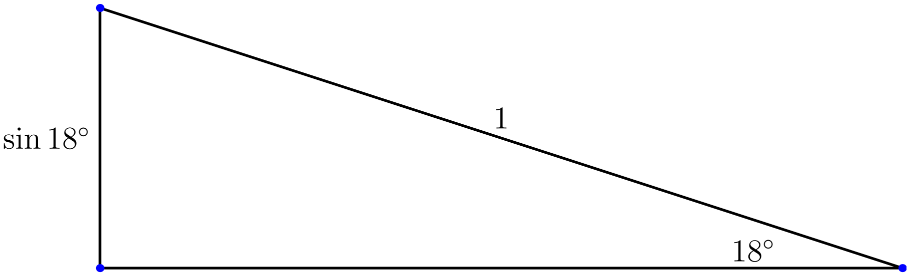
$$1 + 2\cos\left(\frac{2\pi}{5}\right) + 2\cos\left(\frac{4\pi}{5}\right) = 0$$

$$1 + 2t + 2(2t^2 - 1) = 0$$

$$4t^2 + 2t - 1 = 0$$

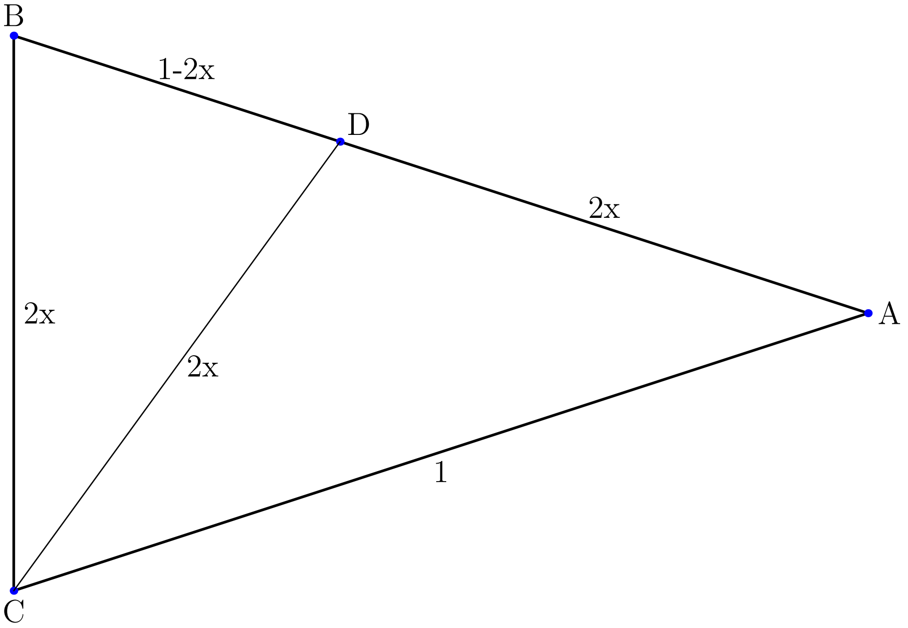
Taking the positive solution, we discover $t = \frac{\sqrt{5}-1}{4}$. Therefore $m + n + k = \boxed{10}$.

Solution (Ayush Rudra).



Note that $\sin\left(\frac{\pi}{10}\right) = \sin(18^\circ)$.

Geometrically, $\sin(18^\circ)$ is the length of the leg opposite of an 18° angle in a right triangle with a hypotenuse of length 1 as shown above. Let $x = \sin(18^\circ)$. Consider the following diagram created by reflecting the triangle across its long leg. D is the angle bisector of $\angle C$. The angles can be found through some basic angle chasing.



Note that triangles $\triangle BCD$ and $\triangle ACD$ are also isosceles. Thus $AD = CD = BC = 2x$ and $BD = 1 - 2x$. Finally, $\triangle ABC \sim CDB$ so $\dfrac{1}{2x} = \dfrac{2x}{1-2x}$. Thus, $4x^2 + 2x - 1 = 0$, or $x = \dfrac{-1 \pm \sqrt{5}}{4}$. Looking at the answer format or using the fact that $x = \sin(18^\circ) > 0$, we can disregard the negative root. Thus, $\sin\left(\frac{\pi}{10}\right) = \sin(18^\circ) = x = \dfrac{\sqrt{5}-1}{4}$, meaning $(m, n, k) = (5, 1, 4)$, so $m + n + k = \boxed{10}$.

Problem 22

Solution (Jack Whitney-Epstein).

Let $n^2 + 11n + 121 = m^2$. We can rearrange as follows:

$$4n^2 + 44n + 484 = 4m^2$$

$$(2n + 11)^2 + 363 = (2m)^2$$

$$(2m)^2 - (2n + 11)^2 = 363.$$

The equation $a^2 - b^2 = 363$ factors as $(a+b)(a-b) = 363 = 3 \cdot 11^2$. This has solutions

$$(a+b),(a-b) = (363,1),(121,3),(33,11)$$

and thus

$$(a,b) = (182,181),(62,59),(22,11).$$

Since $b = 2n+11$, the smallest b which yields a positive n is 59, and this gives $n = \frac{59-11}{2} = \boxed{24}$.

Problem 23

Solution (Jack Whitney-Epstein).

Let p_i be the probability of the particle exiting through room 1, given that it's currently in room i.

Then, we have

$$p_1 = \frac{2}{3} + \frac{1}{3}p_2$$

$$p_2 = \frac{1}{3}p_1 + \frac{1}{3}p_3$$

$$p_3 = \frac{1}{2}p_2$$

upon which solving gives $p_1 = \frac{10}{13}$, so $m+n = \boxed{23}$.

Solution (Marguerite Pennington).

We look at different states to find the probability of escaping from Room 1 based on which Room we started in.

Let a be the probability of escaping through Room 1 from Room 1 (note this is what we are looking for). Similarly, we let b be the probability of escaping through Room 1 from Room 2. We let c be the probability of escaping through Room 1 from Room 3.

From Room 1, there are 3 doors: two of them leave the box through Room 1, the third leads to Room 2. Once in Room 2, there is still a probability b that after some number of moves, you still escape from Room 1. Thus,

$$a = \frac{1}{3}(1) + \frac{1}{3}(1) + \frac{1}{3}b = \frac{2}{3} + \frac{1}{3}b$$

Similarly,

$$b = \frac{1}{3}a + \frac{1}{3}c + \frac{1}{3}(0) = \frac{1}{3}a + \frac{1}{3}c$$

$$c = \frac{1}{2}b + \frac{1}{2}(0) = \frac{1}{2}b$$

Now just solve the 3 equations with 3 variables algebraically. Substitute $c = \frac{1}{2}b$ into the equation for b:

$$b = \frac{1}{3}a + \frac{1}{3}\left(\frac{1}{2}b\right) = \frac{1}{3}a + \frac{1}{6}b$$

We get $\frac{5}{6}b = \frac{1}{3}a$ So $b = \frac{2}{5}a$

Substitute this into the equation for a:

$$a = \frac{2}{3} + \frac{1}{3}\left(\frac{2}{5}a\right) = \frac{2}{3} + \frac{2}{15}a$$

Thus

$$\frac{13}{15}a = \frac{2}{3},$$

which gives

$$a = \frac{2}{3} \cdot \frac{15}{13} = \frac{10}{13}$$

, the probability of escaping through Room 1 from Room 1.

$$m + n = \boxed{23}$$

Problem 24

Solution (Alex Svoronos).

First, notice that $|z_1^2+z_1z_2+z_2^2| \leq 1$ is equivalent to $|4z_1^2+4z_1z_2+4z_2^2| \leq 4$, and $|z_1^2+4z_1z_2+4z_2^2| \leq 1$ is equivalent to $|z_1+2z_2| \leq 1$. So, now, by Triangle Inequality,

$$|3z_1^2| \leq |4z_1^2 + 4z_1z_2 + 4z_2^2| + |z_1^2 + 4z_1z_2 + 4z_2^2| = 5,$$

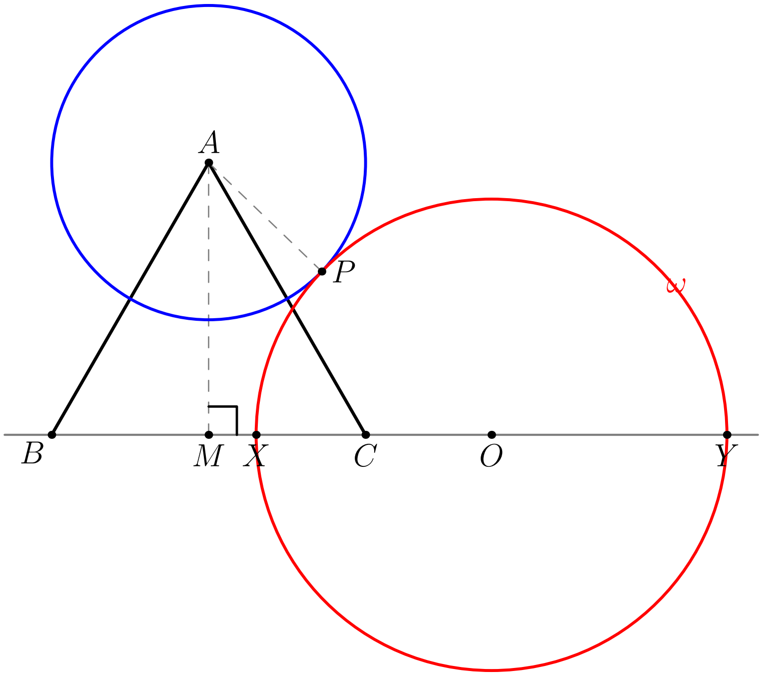
so $|z_1| \leq \frac{\sqrt{15}}{3}$. Now,

$$|2z_2| \leq |z_1 + 2z_2| + |z_1| \leq 1 + \frac{\sqrt{15}}{3}.$$

Since $\frac{1+\frac{\sqrt{15}}{3}}{2} \leq \frac{\sqrt{15}}{3}$, we have that the maximum value of $\max(|z_1|, |z_2|)$ is $\frac{\sqrt{15}}{3}$, giving $15 + 3 = \boxed{18}$ as our answer.

Problem 25

Solution (Jack Whitney-Epstein).



Let r be the maximum value of $\frac{PB}{PC}$. Draw the set of all points D such that $\frac{DB}{DC} = r$. This is known to be a circle ω whose center lies on BC, called the Apollonius circle. Let ω have center O and intersect line BC at points X and Y (with B, X, C in that order).

Since P maximizes $\frac{PB}{PC}$, the two circles are tangent at P. Then $\frac{BX}{CX} = r$ and $BX + CX = 2$, so $BX = \frac{2r}{r+1}$ and $CX = \frac{2}{r+1}$. Similarly $BY = \frac{2r}{r-1}$ and $CY = \frac{2}{r-1}$. So, the radius of ω is $\frac{1}{2}(CX + CY) = \frac{1}{r+1} + \frac{1}{r-1}$. Thus $AO = 1 + \frac{1}{r+1} + \frac{1}{r-1}$. Also, $CO = OX - CX = \frac{1}{r-1} - \frac{1}{r+1}$.

Let M be the midpoint of BC, and consider right triangle AMO. Its lengths are $AM = \sqrt{3}, MO = 1 + \frac{1}{r-1} - \frac{1}{r+1}$, and $AO = 1 + \frac{1}{r+1} + \frac{1}{r-1}$.

We can now use the Pythagorean theorem to solve for r:

$$AM^2 + MO^2 = AO^2$$

$$3 + \left(1 + \frac{1}{r-1} - \frac{1}{r+1}\right)^2 = \left(1 + \frac{1}{r-1} + \frac{1}{r+1}\right)^2$$

$$3 = 4\left(\frac{1}{r+1}\right)\left(1 + \frac{1}{r-1}\right)$$

$$3r^2 - 4r - 3 = 0$$

Solve the quadratic to get $r = \frac{2+\sqrt{13}}{3}$, for an answer of $\boxed{18}$.