CT ARML Team, 2026
Team Selection Test 1

1. Let $A = \{1, 2, 3, \ldots, 100\}$ and $B = \{a^2 + 2^a \mid a \in A\}$. Find the number of elements in $A \cup B$.

2. There is exactly one way to write 2026 as a sum of at least two consecutive positive integers: $2026 = a + (a+1) + \cdots + (b-1) + b$. Compute $b - a$.

3. Let
$$P = (\log_2 9)(\log_3 16)(\log_4 25) \cdots (\log_{999} 1000000).$$
Find $\lfloor \log_2 P \rfloor$.
(Note: $\lfloor x \rfloor$, the *floor* of x, is the greatest integer less than or equal to x.)

4. Let $f(x)$ be a function with domain $\mathbb{R}$ that is periodic with minimum positive period equal to 5. Suppose that the number of zeros of the function $g(x) = f(2^x)$ in the interval $[0, 5)$ is 25. Compute the number of zeros of $g(x)$ in the interval $[1, 4)$.
(Note: The *minimum positive period* of a periodic function f is the smallest positive number p such that $f(x + p) = f(x)$ for all real numbers x.)

5. Let
$$S_n = \sum_{k=0}^{n} \arctan\left(\frac{1}{k^2 + k + 1}\right)$$
Compute $\tan(S_{2026})$.

6. Suppose that $\triangle ABD$ has C on $\overline{BD}$, with C between B and D. Also, $BC = 2$, $CD = 1$, $\frac{AC}{AD} = \frac{3}{4}$, and $\cos(\angle ACD) = -\frac{3}{5}$. Then, $AB = \frac{p}{q}$, where p and q are relatively prime positive integers. Compute $p + q$.

7. The year is 1183. A time traveler can jump into the future or the past only by a number of years that is a divisor of 1183. He wants to attend an event that took place in 1059. What is the smallest number of jumps that he should take? (We assume that there is no waiting time between successive jumps.)

8. Let $ABCD$ be a parallelogram and let E be a point on ray $\overrightarrow{AD}$ with D between A and E. Segment $\overline{BE}$ intersects $\overline{AC}$ at F and $\overline{CD}$ at G. If $BF = EG$ and $BC = 3$, then $AE = \frac{p+\sqrt{q}}{r}$, where p, q, r are positive integers and p and r are relatively prime. Compute $p + q + r$.

9. Let
$$z = \sum_{k=1}^{13} \frac{1}{1 - e^{ki\pi/7}}.$$
Then $|z| = \frac{p}{q}$, where p and q are relatively prime positive integers. Compute $p + q$.

10. Let $a_1, a_2, \ldots, a_{35}$ be positive integers such that for any k with $1 \leq k \leq 35$,
$$a_k = \sum_{i=1}^{35} |a_k - a_i|.$$

Find the minimum possible value of $a_1 + a_2 + \cdots + a_{35}$.

CT ARML Team, 2026
Team Selection Test 1
Answers

1. 194
2. 3
3. 1001
4. 11
5. 2027
6. 20
7. 4
8. 50
9. 15
10. 612

CT ARML Team, 2026
Team Selection Test 1
Solutions

1. Let $A = \{1, 2, 3, \ldots, 100\}$ and $B = \{a^2 + 2^a \mid a \in A\}$. Find the number of elements in $A \cup B$.
[Answer: 194]

Solution (Vikram Sarkar)
Note that since $6^2+2^6 = 100$, and x^2+2^x is increasing, we have that for $1 \le x \le 6$, $x^2+2^x \le 100$, and for $x > 7$, $x^2 + 2^x > 100$. Therefore, $|A \cap B| = 6$. Thus,

$$|A \cup B| = |A| + |B| - |A \cap B| = 100 + 100 - 6 = \boxed{194}.$$

2. There is exactly one way to write 2026 as a sum of at least two consecutive positive integers: $2026 = a + (a + 1) + \cdots + (b - 1) + b$. Compute $b - a$.
[Answer: 3]

Solution (Vikram Sarkar)
Note that by the arithmetic sequence sum formula,

$$2026 = \frac{a+b}{2}(b - a + 1),$$

and so $(a+b)(b-a+1) = 4052$. Note that $a+b$ and $b-a+1$ are of different parities (one even, one odd). Furthermore, $b + a > b - a + 1$. Therefore, the possible options for $(a + b, b - a + 1)$ are $(1013, 4)$ and $(4052, 1)$, as 1013 is prime.
If $a + b = 1013$ and $b - a + 1 = 4$, then since $b - a = 3$, we have that by adding, $b = 508$, and so $a = 505$.
If $a+b = 4052$ and $b-a+1 = 1$, then $b = a$, however then there would be only one consecutive positive integer.
So we have $(a, b) = (505, 508)$ and so $b - a = \boxed{3}$.

Alternative Solution (Girish Prasad)
Say we can write 2026 as the sum of $n > 1$ consecutive positive integers.

Notice that if n is odd, the series of positive integers must have a center term equal to $\frac{2026}{n}$. Hence, n would have to be a factor of 2026. However, since $n > 1$, this means $n = 1013$, in which case the center term would be 2 and the series would have negative integers.

Therefore, n must be even. In this case, the average value of all the positive integers in the series, $\frac{2026}{n}$, must be equal to half of an odd positive integer. We can quickly notice that $n = 4$ satisfies this criteria and the respective series is $505 + 506 + 507 + 508 = 2026$. Hence, $b - a = 508 - 505 = \boxed{3}$.

3. Let

$$P = (\log_2 9)(\log_3 16)(\log_4 25) \cdots (\log_{999} 1000000).$$

Find $\lfloor \log_2 P \rfloor$.
(Note: $\lfloor x \rfloor$, the *floor* of x, is the greatest integer less than or equal to x.)
[Answer: 1001]

Solution (Vikram Sarkar)
Note that

$$\begin{aligned} P &= \log_2(3^2)\log_3(4^2)\cdots\log_{999}(1000^2) \\ &= (2\log_2 3)(2\log_3 4)\cdots(2\log_{999}(1000)) \\ &= 2^{998}(\log_2(3)\log_3(4)\cdots\log_{999}(1000)). \end{aligned}$$

Remember the "logarithm chain rule" (useful to know):

$$\log_a b \cdot \log_b c = \log_a c.$$

Thus this product $\log_2 3 \cdot \log_3 4 \cdots \log_{999}(1000)$ telescopes to $\log_2(1000)$. Therefore, $P = 2^{998}\log_2(1000)$. Consequently, $\log_2(P) = 998 + \log_2(\log_2(1000))$. Since $\log_2(1000) \in (9, 10)$, $\log_2(\log_2(1000)) \in (3, 4)$. Thus

$$\lfloor \log_2(P) \rfloor = 998 + \lfloor \log_2(\log_2(1000)) \rfloor = 998 + 3 = \boxed{1001}.$$

Note: Here's a smaller example to illustrate the "telescoping":

$$[\log_2(3)\log_3(4)]\log_4(5)\log_5(6) = [\log_2(4)\log_4(5)]\log_5(6) = \log_2(5)\log_5(6) = \log_2(6).$$

4. Let $f(x)$ be a function with domain $\mathbb{R}$ that is periodic with minimum positive period equal to 5. Suppose that the number of zeros of the function $g(x) = f(2^x)$ in the interval $[0, 5)$ is 25. Compute the number of zeros of $g(x)$ in the interval $[1, 4)$.
(Note: The *minimum positive period* of a periodic function f is the smallest positive number p such that $f(x + p) = f(x)$ for all real numbers x.)
[Answer: 11]

Solution (Vikram Sarkar)
Note that if $x \in [0, 5)$, then $2^x \in [1, 32)$ and can achieve everything in this range. Thus, g having 25 zeroes in the interval $[0, 5)$ is equivalent to f having 25 zeroes in the interval $[1, 32)$. Now, split up

$$[1, 32) = [1, 6) \cup [6, 11) \cup [11, 16) \cup [16, 21) \cup [21, 26) \cup [26, 31) \cup [31, 32).$$

Each of these intervals besides the last one is of the form $[5k + 1, 5k + 6)$ for some integer k. Since f has period 5, the number of f-zeroes in each of these intervals is the same as the number of f-zeroes in $[1, 6)$, since $f(5k + x) = f(x)$. So, let

$$A = \# \text{ of } f - \text{zeroes in } [1, 6).$$

Now, by similar logic, the number of f-zeroes in $[31, 32)$ is the same number of f-zeroes as in $[1, 2)$, since $31 \equiv 1 \pmod 5$. Therefore, if we let

$$B = \# \text{ of } f - \text{zeroes in } [1, 2),$$

then we have

$$6A + B = 25.$$

Clearly $A \geq B$, since the interval $[1, 2)$ is a subset of $[1, 6)$. Thus it is clear that we must have $(A, B) = (4, 1)$. Now, we want to find the number of g-zeroes in $[1, 4)$, which is the same as the number of f-zeroes in $[2, 16)$. The number of f-zeroes in $[2, 16)$ is one less than the number of f-zeroes in $[1, 16)$, since the number of f-zeroes in $[1, 2)$ is $B = 1$. Now, split up

$$[1, 16) = [1, 6) \cup [6, 11) \cup [11, 16),$$

and so there are $3A = 12$ zeroes $[1, 16)$. Therefore the number of f-zeroes in $[2, 16)$ is $12 - 1 = \boxed{11}$.

5. Let

$$S_n = \sum_{k=0}^{n} \arctan\left(\frac{1}{k^2 + k + 1}\right)$$

Compute $\tan(S_{2026})$.
[Answer: 2027]

Solution (Vikram Sarkar)
The main idea is that for all $k \geq 1$, we have that

$$\arctan\left(\frac{1}{k^2 + k + 1}\right) = \arctan\left(\frac{1}{k}\right) - \arctan\left(\frac{1}{k+1}\right).$$

This is because, by the tangent addition formula,

$$\begin{aligned}
\tan\left(\arctan\left(\frac{1}{k}\right) - \arctan\left(\frac{1}{k+1}\right)\right) &= \frac{\tan\left(\arctan\left(\frac{1}{k}\right)\right) - \tan\left(\arctan\left(\frac{1}{k+1}\right)\right)}{1 + \tan\left(\arctan\left(\frac{1}{k}\right)\right)\tan\left(\arctan\left(\frac{1}{k+1}\right)\right)} \\
&= \frac{\frac{1}{k} - \frac{1}{k+1}}{1 + \frac{1}{k^2+k}} \\
&= \frac{1}{k^2 + k + 1},
\end{aligned}$$

and since $\arctan\left(\frac{1}{k}\right) - \arctan\left(\frac{1}{k+1}\right) \in \left(0, \frac{\pi}{2}\right)$. Therefore, we can compute

$$\begin{aligned}
S_{2026} &= \sum_{k=0}^{2026} \arctan\left(\frac{1}{k^2 + k + 1}\right) \\
&= \arctan(1) + \sum_{k=1}^{2026} \arctan\left(\frac{1}{k^2 + k + 1}\right) \\
&= \arctan(1) + \sum_{k=1}^{2026} \left(\arctan\left(\frac{1}{k}\right) - \arctan\left(\frac{1}{k+1}\right)\right) \\
&= \arctan(1) + \arctan(1) - \arctan\left(\frac{1}{2027}\right) \\
&= \frac{\pi}{2} - \arctan\left(\frac{1}{2027}\right) \\
&= \arctan(2027),
\end{aligned}$$

and so $\tan(S_{2026}) = \boxed{2027}$.

Alternative Solution (Anay Sahu)

We prove by induction that $S_n = \tan^{-1}(n+1)$. As the base case, when $n = 0$, we get $S_0 = \tan^{-1}(1)$ and the claim holds. For the inductive step, assume that for some integer $n \geq 1$, $S_{n-1} = \tan^{-1}(n)$. By the definition of S_n,

$$\begin{aligned} S_n &= S_{n-1} + \tan^{-1}\left(\frac{1}{n^2+n+1}\right) \\ &= \tan^{-1}(n) + \tan^{-1}\left(\frac{1}{n^2+n+1}\right) \end{aligned}$$

Using the identity $\tan^{-1}(a) + \tan^{-1}(b) = \tan^{-1}\left(\frac{a+b}{1-ab}\right)$, we get

$$\begin{aligned} S_n &= \tan^{-1}\left(\frac{n + \frac{1}{n^2+n+1}}{1 - \frac{n}{n^2+n+1}}\right) \\ &= \tan^{-1}\left(\frac{n(n^2+n+1)+1}{n^2+1}\right) \\ &= \tan^{-1}\left(\frac{n^3+n^2+n+1}{n^2+1}\right) \\ &= \tan^{-1}(n+1) \end{aligned}$$

Hence, $S_{2026} = \tan^{-1}(2027)$, and $\tan(S_{2026}) = \boxed{2027}$.

6. Suppose that $\triangle ABD$ has C on $\overline{BD}$, with C between B and D. Also, $BC = 2$, $CD = 1$, $\frac{AC}{AD} = \frac{3}{4}$, and $\cos(\angle ACD) = -\frac{3}{5}$. Then, $AB = \frac{p}{q}$, where p and q are relatively prime positive integers. Compute $p+q$.
[Answer: 20]

Solution (Vikram Sarkar)

Let $AC = 3x$ and $AD = 4x$. Then, by the Law of Cosines on $\triangle ACD$,

$$(4x)^2 = 1^2 + (3x)^2 - 2(3x)\cos\angle ACD = 9x^2 + \frac{18}{5}x + 1,$$

and so

$$7x^2 - \frac{18}{5}x - 1 = 0,$$

and so $35x^2 - 18x - 5 = 0$. This factors as $(7x-5)(5x+1) = 0$, and so $x = \frac{5}{7}$. Therefore, $AC = \frac{15}{7}$. Now, note that

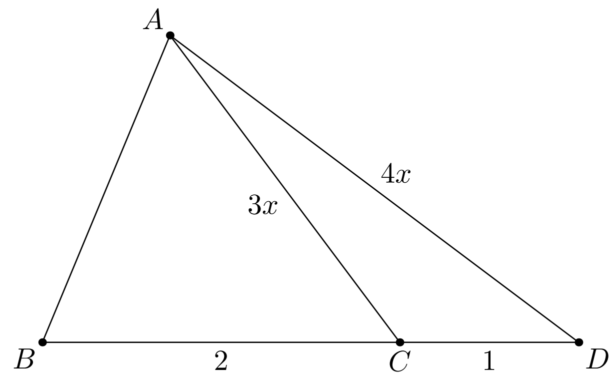
$$\cos\angle ACB = \cos(180^\circ - \angle ACD) = -\cos\angle ACD = \frac{3}{5}.$$

Therefore, by the Law of Cosines in $\triangle ABC$,

$$AB^2 = 2^2 + \left(\frac{15}{7}\right)^2 - 2\cdot 2\cdot \frac{15}{7}\cdot\frac{3}{5} = \frac{169}{49},$$

and so $AB = \frac{13}{7}$, yielding $13 + 7 = \boxed{20}$.

Alternative Solution (Anay Sahu)



Let $AC = 3x$ and $AD = 4x$. Using the Law of Cosines on $\triangle ACD$, we get

$$(4x)^2 = (3x)^2 + 1 - 2(3x)\left(-\frac{3}{5}\right)$$
$$35x^2 - 18x - 5 = 0$$
$$(5x+1)(7x-5) = 0$$
$$x = -\frac{1}{5}, \frac{5}{7}$$

Taking the positive root, we get $AC = \frac{15}{7}$. Since $\cos(\angle ACB) = \cos(180 - \angle ACD) = \frac{3}{5}$, we can use Law of Cosines on $\triangle ACB$ to get

$$AB^2 = 4 + \left(\frac{15}{7}\right)^2 - 2(2)\left(\frac{15}{7}\right)\left(\frac{3}{5}\right)$$
$$AB^2 = \frac{169}{49}$$
$$AB = \frac{13}{7}$$

so our answer is $13 + 7 = \boxed{20}$.

7. The year is 1183. A time traveler can jump into the future or the past only by a number of years that is a divisor of 1183. He wants to attend an event that took place in 1059. What is the smallest number of jumps that he should take? (We assume that there is no waiting time between successive jumps.)
[Answer: 4]

Solution (Vikram Sarkar)
Note that $1183 = 7 \cdot 13^2$, and that $1183 - 1059 = 124$. Therefore it is equivalent to asking what's the minimum number of jumps to get from 0 to 124 where each jump size is $\{\pm 1, \pm 7, \pm 13, \pm 91, \pm 169, \pm 1183\}$. First, note that

$$91 + 13 + 13 + 7 = 124$$

is a way with 4 jumps. Clearly we are not using 1183 since the absolute value of the number after four jumps using 1183 is at least $1183 - 3 \cdot 169 = 676 > 124$. If we use 169, we need to get to $169 - 124 = 45$ in at most three jumps, which is impossible since 91 is too big, and $13 \cdot 3 = 39 < 45$. If we don't use a 91, then we need at least $\lfloor 124/13 \rfloor = 9$ jumps. So we must use a 91, and so we need to get to $124 - 91 = 33$ in at most 3 jumps. $13 + 13 + 7$ works as mentioned before, and $13 \cdot 2 = 26$ is too small. So the minimum is $\boxed{4}$ jumps.

8. Let $ABCD$ be a parallelogram and let E be a point on ray $\overrightarrow{AD}$ with D between A and E. Segment $\overline{BE}$ intersects $\overline{AC}$ at F and $\overline{CD}$ at G. If $BF = EG$ and $BC = 3$, then $AE = \frac{p+\sqrt{q}}{r}$, where p, q, r are positive integers and p and r are relatively prime. Compute $p + q + r$.
[Answer: 50]

Solution (Vikram Sarkar)
Let the line through F parallel to AB and CD intersect side BC at H. Then, since $EG = FB$ and $\angle DEG = \angle FBH$ as $AD \parallel BC$, we have that $DG = FH$. Let this common value be x, and let $AB = CD = y$. Then, $GC = y - x$. Now, let h_1 be the distance from line AB to FH and let h_2 be the distance from line FH to CD. Since $\triangle BFH \sim \triangle BGC$,

$$\frac{x}{y-x} = \frac{h_1}{h_1 + h_2}.$$

Similarly, since $\triangle CFH \sim \triangle CAB$,

$$\frac{x}{y} = \frac{h_2}{h_1 + h_2}.$$

Adding these two equations yields

$$\frac{x}{y-x} + \frac{x}{y} = 1.$$

Now, let $r = \frac{y}{x}$. Then $\frac{x}{y-x} = \frac{1}{\frac{y}{x}-1} = \frac{1}{r-1}$, and $\frac{x}{y} = \frac{1}{r}$. Then,

$$\frac{1}{r-1} + \frac{1}{r} = 1,$$

so

$$\frac{1}{r-1} = 1 - \frac{1}{r} = \frac{r-1}{r},$$

and so

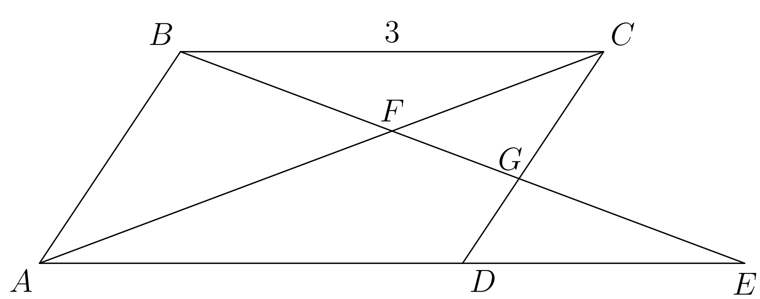
$$r = (r-1)^2.$$

Solving yields $r = \frac{\sqrt{5}+3}{2}$. Now, let $AE = t$, then $ED = t - 3$. Since $\triangle AEB \sim \triangle DEG$, we have

$$\frac{t}{t-3} = \frac{EA}{ED} = \frac{y}{x} = \frac{\sqrt{5}+3}{2}.$$

Thus $t = (t-3)\frac{\sqrt{5}+3}{2}$, and solving yields $t = \frac{3+3\sqrt{5}}{2} = \frac{3+\sqrt{45}}{2}$, which gives $3 + 45 + 2 = \boxed{50}$.

Alternative Solution (Girish Prasad)



Since $AD = 3$, we can find the length of $AE = AD + DE$ by finding the length of DE. Let L be the length of DE.

From pairs of similar triangles, namely $\triangle EDG \sim \triangle BCG$ and $\triangle EAF \sim \triangle BCF$, it follows that

$$\frac{L}{3} = \frac{EG}{GB}$$
$$\frac{3}{L+3} = \frac{BF}{FE}$$

Since $EG = BF$ and $GB = FE$, it follows that

$$\frac{L}{3} = \frac{3}{L+3}$$
$$L(L+3) = 9$$
$$L^2 + 3L - 9 = 0$$

Since $L > 0$,

$$L = \frac{-3+\sqrt{45}}{2}$$
$$AE = 3 + L = \frac{3+\sqrt{45}}{2}$$

Therefore, the final answer is $3 + 45 + 2 = \boxed{50}$.

9. Let

$$z = \sum_{k=1}^{13} \frac{1}{1 - e^{ki\pi/7}}.$$

Then $|z| = \frac{p}{q}$, where p and q are relatively prime positive integers. Compute $p + q$. [Answer: 15]

Solution
Factoring $e^{ik\pi/14}$ from the denominator,

$$1 - e^{ik\pi/7} = e^{ik\pi/14}\left(e^{-ik\pi/14} - e^{ik\pi/14}\right) = -2i\sin\left(\tfrac{k\pi}{14}\right) e^{ik\pi/14}.$$

Therefore

$$\frac{1}{1-e^{ik\pi/7}} = \frac{e^{-ik\pi/14}}{-2i\sin(k\pi/14)} = \frac{\cos(k\pi/14) - i\sin(k\pi/14)}{-2i\sin(k\pi/14)} = \frac{1}{2} + \frac{i}{2}\cot\left(\tfrac{k\pi}{14}\right).$$

Summing from $k = 1$ to 13,

$$z = \frac{13}{2} + \frac{i}{2}\sum_{k=1}^{13}\cot\left(\tfrac{k\pi}{14}\right).$$

The cotangent sum vanishes: $\cot(7\pi/14) = 0$, and since $\cot(\pi - x) = -\cot x$, the remaining terms pair off as $\cot(\pi/14) + \cot(13\pi/14) = 0$, $\cot(2\pi/14) + \cot(12\pi/14) = 0$, and so on through

$\cot(6\pi/14) + \cot(8\pi/14) = 0$. Hence $z = \frac{13}{2}$, so $|z| = \frac{13}{2}$, and $p + q = \boxed{15}$.

Alternative Solution (Vikram Sarkar)
Let $P(z) = \prod_{k=1}^{13}(z - e^{k\pi i/7}) = \frac{z^{14}-1}{z-1} = z^{13} + z^{12} + \cdots + z + 1$, since $e^{k\pi i/7}$ are the fourteenth roots of unity, except we missed out $k = 0$, corresponding to 1. Now, suppose the roots of P are $r_1, r_2, \cdots, r_{13}$, i.e. $P(z) = (z - r_1)(z - r_2)\cdots(z - r_{13})$. Then, we want

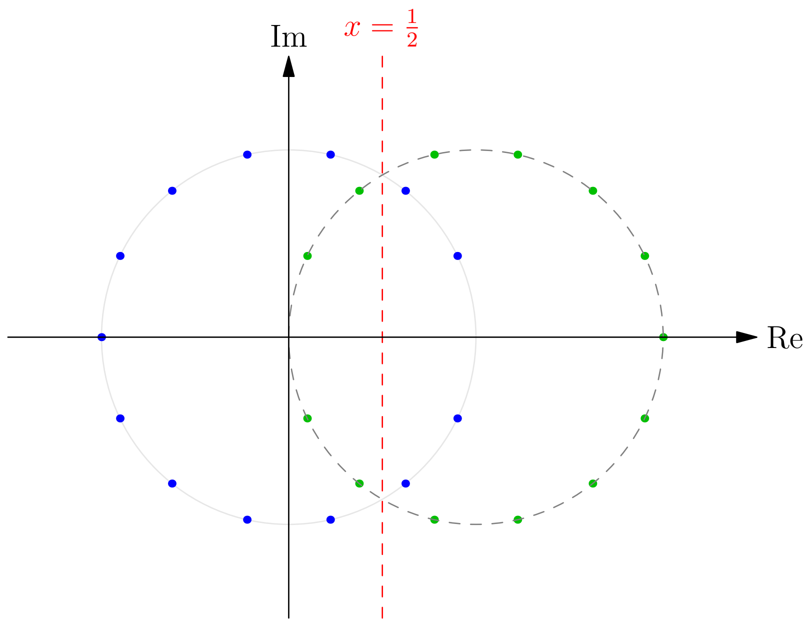
$$\begin{aligned}
\sum_{k=1}^{13} \frac{1}{1 - r_k} &= \frac{\sum_{k=1}^{13} \prod_{j \neq k}(1 - r_j)}{(1 - r_1)(1 - r_2)\cdots(1 - r_{13})} \\
&= \frac{\frac{\mathrm{d}}{\mathrm{d}x}((x - r_1)(x - r_2)\cdots(x - r_{13}))\Big|_{x=1}}{P(1)} \\
&= \frac{P'(1)}{P(1)} \\
&= \frac{13 + 12 + \cdots + 1}{14} \\
&= \frac{13}{2},
\end{aligned}$$

which gives $13 + 2 = \boxed{15}$.
To spell it out more, suppose we had three roots r_1, r_2, r_3. Then,

$$\frac{1}{1 - r_1} + \frac{1}{1 - r_2} + \frac{1}{1 - r_3} = \frac{(1 - r_1)(1 - r_2) + (1 - r_1)(1 - r_3) + (1 - r_2)(1 - r_3)}{(1 - r_1)(1 - r_2)(1 - r_3)}.$$

Now, consider $P(x) = (x - r_1)(x - r_2)(x - r_3)$. Clearly the denominator is $P(1)$. Now, consider $P'(x)$. By the product rule, this is $(x-r_1)(x-r_2)+(x-r_1)(x-r_3)+(x-r_2)(x-r_3)$. Thus the numerator is just this evaluated at $x = 1$, i.e., $P'(1)$. This is a very useful trick worth knowing as it shows up fairly often.

Alternative Solution (Ayush Rudra)



Geometrically, the points $e^{ki\pi/7}$ for $1 \leq k \leq 13$ are the 14th roots of unity excluding $z = 1$, as shown in blue above. They satisfy $z^{14} = 1$ and lie on the unit circle centered at the origin of the complex plane. Similarly, $1 - e^{ki\pi/7}$ is just the same diagram, reflected over $x = \dfrac{1}{2}$. This is shown above with the green points. In the complex plane, the function $\dfrac{1}{z}$ is the same as doing an inversion through the unit circle and then reflecting that result over the real axis. An inversion of a circle with center O through a point on that circle C is a line perpendicular to $\overline{OC}$. So, $\dfrac{1}{z}$ will send the circle with the green points on it to a line l perpendicular to the real axis. The point 2 on this circle gets mapped to $\dfrac{1}{2}$, so our line l is $x = \dfrac{1}{2}$. If we take two green points that are complex conjugates and apply the mapping $\dfrac{1}{z}$, they will still be complex conjugates. So, all our green points map to points on the line $x = \dfrac{1}{2}$ with a symmetry over the real axis. Thus, $z = \sum_{k=1}^{13} \dfrac{1}{1 - e^{ki\pi/7}} = 13 \cdot \dfrac{1}{2} = \dfrac{13}{2}$. This means $|z| = \dfrac{13}{2}$, so our answer is $13 + 2 = \boxed{15}$.

10. Let $a_1, a_2, \ldots, a_{35}$ be positive integers such that for any k with $1 \leq k \leq 35$,

$$a_k = \sum_{i=1}^{35} |a_k - a_i|.$$

Find the minimum possible value of $a_1 + a_2 + \cdots + a_{35}$.
[Answer: 612]

Solution (Vikram Sarkar)
Without loss of generality, suppose $a_1 \leq a_2 \leq \cdots \leq a_{35}$. Obviously all a_i are not all equal otherwise they would all be zero. So suppose $a_1 = a_2 = \cdots a_m = a$ for some $1 \leq m \leq 34$ and

suppose $a_{m+1} = b > a$. Then, we have that

$$a = a_1 = \sum_{i=1}^{35} |a_i - a| \geq \sum_{i=m+1}^{35} |a_i - a| \geq \sum_{i=m+1}^{35} (b-a) = (35-m)(b-a),$$

as $a_i \geq a_{m+1} = b$ for all $i \geq m+1$. Furthermore, we have that

$$b = a_{m+1} = \sum_{i=1}^{35} |a_i - b| \geq \sum_{i=1}^{m} |a_i - b| = m(b-a).$$

Adding these two equations, we have that

$$a + b \geq 35(b-a),$$

which rearranges to $18a \geq 17b$. Therefore, $b \leq \frac{18}{17}a$. Since $b > a$, $b \geq a+1$. Thus, $\frac{18}{17}a \geq a+1$, and so $a \geq 17$.
If $a = 17$, then $b \leq 18$ and $b > a = 17$, so $b = 18$. Plugging this into our two inequalities, $17 \geq 35 - m$ and $18 \geq m$. Therefore, $m = 18$. Thus,

$$\sum_{i=1}^{35} a_i = \sum_{i=1}^{18} a_i + \sum_{i=19}^{35} a_i \geq 18a + 17b = 2 \cdot 17 \cdot 18 = 612.$$

Equality holds when

$$a_n = \begin{cases} 17 & n \leq 18 \\ 18 & n > 18. \end{cases}$$

If $a > 17$, then $\sum_{i=1}^{35} a_i \geq 35a \geq 35 \cdot 18 = 630$, which is nonoptimal. So the minimum is $\boxed{612}$.
Remark: Replace 35 with n. Then, using some linear algebra, we can determine the following: If n is even, and $a_k = \sum_{i=1}^{n} |a_k - a_i|$ for all $1 \leq i \leq n$ (and a_i need not be integers, just reals, and we forget about the minimum possible value of $\sum a_i$, we just want to determine all such $(a_1, \cdots, a_n)$), then we must have $a_1 = a_2 = \cdots = a_n = 0$.
If n is odd, and we have the same condition, then we must have $(a_1, a_2, \cdots, a_n)$ is some permutation of $\left(c\left(\frac{n-1}{2}\right), c\left(\frac{n-1}{2}\right), \cdots, c\left(\frac{n-1}{2}\right), c\left(\frac{n+1}{2}\right), \cdots, c\left(\frac{n+1}{2}\right)\right)$, where there are $\frac{n+1}{2}$ of $c\left(\frac{n-1}{2}\right)$ and $\frac{n-1}{2}$ of $c\left(\frac{n+1}{2}\right)$, and c is any real number.

9 Relay Problems

Relay 1-1. The number $N = 327763$ can be expressed as

$$N = 51^3 + 58^3 = 30^3 + 67^3.$$

Compute the least prime factor of N.

Relay 1-2. Let $T = TNYWR$. Rectangle $ABCD$ has area $T + 5$. Point M is the midpoint of $\overline{AB}$ and point N is a trisection point on $\overline{CD}$. The segment $\overline{MN}$ divides rectangle $ABCD$ into two trapezoids. Compute the area of the larger of these two trapezoids.

Relay 1-3. Let $T = TNYWR$. Define the sequence $a_1, a_2, a_3, \ldots$ by $a_1 = \sqrt[4]{2}, a_2 = \sqrt{2}$, and for $n \geq 3$, $a_n = a_{n-1}a_{n-2}$. Compute the least value of k such that a_k is an integer multiple of $2^{\lfloor T \rfloor}$.

10 Relay Answers

Answer 1-1. 31

Answer 1-2. 21

Answer 1-3. 11

11 Relay Solutions

Relay 1-1. The number $N = 327763$ can be expressed as

$$N = 51^3 + 58^3 = 30^3 + 67^3.$$

Compute the least prime factor of N.

Solution 1-1. Note that the factorization $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ implies that $a + b$ is a divisor of $a^3 + b^3$. Thus $51 + 58 = 109$ and $30 + 67 = 97$ are each factors of N. From long division, $N = 31 \cdot 97 \cdot 109$, and each of the three factors in the product is prime, hence the answer is **31**.

Alternate Solution: Note that $N = 327 \cdot 1000 + 763$ and that $327 = 3 \cdot 109$ and $763 = 7 \cdot 109$. Thus $N = 109(3 \cdot 1000 + 7)$. As in the first solution, conclude that 97 must be a factor of 3007 and, using long division, the other factor is 31.

Note: Numbers that can be represented as the sum of two positive cubes in two different ways are examples of *taxicab numbers*. The least taxicab number is 1729 ($1729 = 10^3 + 9^3 = 12^3 + 1^3$). The origin of the name "taxicab number" comes from an anecdote involving a visit between mathematicians G. H. Hardy and S. Ramanujan. Hardy related that the number of the taxi he took to visit Ramanujan was 1729, and he remarked that the number seemed to be rather dull. Ramanujan countered by stating the fact discussed above.

Relay 1-2. Let $T = TNYWR$. Rectangle $ABCD$ has area $T + 5$. Point M is the midpoint of $\overline{AB}$ and point N is a trisection point on $\overline{CD}$. The segment $\overline{MN}$ divides rectangle $ABCD$ into two trapezoids. Compute the area of the larger of these two trapezoids.

Solution 1-2. Let $AB = CD = 6x$ and $BC = AD = y$. The smaller trapezoid has bases of lengths $2x$ and $3x$, so its area is $\frac{5}{2}xy$. The larger trapezoid has bases of lengths $3x$ and $4x$, so its area is $\frac{7}{2}xy$. Because $6xy = T + 5$, the area of the larger trapezoid is $\frac{7}{12}(T + 5)$. With $T = 31$, the desired area is **21**.

Relay 1-3. Let $T = TNYWR$. Define the sequence $a_1, a_2, a_3, \ldots$ by $a_1 = \sqrt[4]{2}, a_2 = \sqrt{2}$, and for $n \geq 3$, $a_n = a_{n-1}a_{n-2}$. Compute the least value of k such that a_k is an integer multiple of $2^{\lfloor T \rfloor}$.

Solution 1-3. Make a table consisting of the first few values of the sequence $a_1, a_2, a_3, \ldots$.

n	1	2	3	4	5	6
a_n	$2^{1/4}$	$2^{2/4}$	$2^{3/4}$	$2^{5/4}$	$2^{8/4}$	$2^{13/4}$

Note that a_n is of the general form $2^{F_{n+1}/4}$, where F_n denotes the n^{th} Fibonacci number. The desired value of k has the property that F_{k+1} is a multiple of 4 and is the least such value satisfying $F_{k+1} \geq 4\lfloor T \rfloor$. With $T = 21$, the least Fibonacci number that is both a multiple of 4 and which is greater than or equal to $4 \cdot 21 = 84$ is $F_{12} = 144$, hence $k = \mathbf{11}$.

1 Team Problems

Problem 1. Compute the number of non-empty subsets S of $\{1, 2, 3, \ldots, 20\}$ for which $|S| \cdot \max\{S\} = 18$. (Note: $|S|$ is the number of elements of the set S.)

Problem 2. A class of 218 students takes a test. Each student's score is an integer from 0 to 100, inclusive. Compute the greatest possible difference between the mean and the median scores.

Problem 3. Regular hexagon $RANGES$ has side length 6. Pentagon $RANGE$ is revolved $360°$ about the line containing $\overline{RE}$ to obtain a solid. The volume of the solid is $k \cdot \pi$. Compute k.

Problem 4. A fair 12-sided die has faces numbered 1 through 12. The die is rolled twice, and the results of the two rolls are x and y, respectively. Given that $\tan(2\theta) = \frac{x}{y}$ for some θ with $0 < \theta < \frac{\pi}{2}$, compute the probability that $\tan\theta$ is rational.

Problem 5. The absolute values of the five complex roots of $z^5 - 5z^2 + 53 = 0$ all lie between the positive integers a and b, where $a < b$ and $b - a$ is minimal. Compute the ordered pair (a, b).

Problem 6. Let S be the set of points (x, y) whose coordinates satisfy the system of equations:

$$\lfloor x \rfloor \cdot \lceil y \rceil = 20$$
$$\lceil x \rceil \cdot \lfloor y \rfloor = 18.$$

Compute the least upper bound of the set of distances between points in S.

Problem 7. Compute the least integer $d > 0$ for which there exist distinct lattice points A, B, and C in the coordinate plane, each exactly $\sqrt{d}$ units from the origin, satisfying $\csc(\angle ABC) > 2018$.

Problem 8. Compute the number of unordered collections of three integer-area rectangles such that the three rectangles can be assembled without overlap to form one 3×5 rectangle. (For example, one such collection contains one 3×3 and two 1×3 rectangles, and another such collection contains one 3×3 and two 2×1.5 rectangles. The latter collection is equivalent to the collection of two 1.5×2 rectangles and one 3×3 rectangle.)

Problem 9. Let Γ be a circle with diameter $\overline{XY}$ and center O, and let γ be a circle with diameter $\overline{OY}$. Circle ω_1 passes through Y and intersects Γ and γ again at A and B, respectively. Circle ω_2 also passes through Y and intersects Γ and γ again at D and C, respectively. Given that $AB = 1$, $BC = 4$, $CD = 2$, and $AD = 7$, compute the sum of the areas of ω_1 and ω_2.

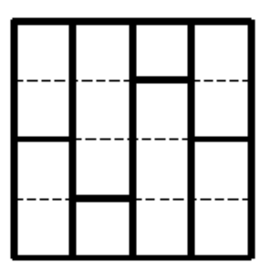
Problem 10. In the number puzzle below, clues are given for the four rows, each of which contains a four-digit number. Cells inside a region bounded by bold lines must all contain the same digit, and each of the eight regions contains a different digit. The variables in the clues are all positive integers. Complete the number puzzle.

1: $4^a + 13^b + 14^c$

2: $5^p + 13^q + 17^r$

3: $4^x + 5^y + 31^z$

4: the average of the other 3 rows



2 Answers to Team Problems

Answer 1. 19

Answer 2. $\dfrac{5400}{109}$ $\left(\text{or } 49\dfrac{59}{109}\right)$

Answer 3. $342\sqrt{3}$

Answer 4. $\dfrac{1}{18}$ (or $0.0\bar{5}$)

Answer 5. $(2, 3)$

Answer 6. $2\sqrt{73}$

Answer 7. 2,038,181

Answer 8. 132

Answer 9. 20π

Answer 10.

5	1	9	7
5	1	0	7
4	1	0	2
4	8	0	2

3 Solutions to Team Problems

Problem 1. Compute the number of non-empty subsets S of $\{1, 2, 3, \ldots, 20\}$ for which $|S| \cdot \max\{S\} = 18$. (Note: $|S|$ is the number of elements of the set S.)

Solution 1. Both $|S|$ and $\max\{S\}$ are integers between 1 and 20, so factor 18 and consider the possible ways to have $|S|$ and $\max\{S\}$ equal each factor. Then count the number of possible sets S that satisfy that condition.

$\lvert S\rvert$	$\max\{S\}$	**Number of possible sets S**
1	18	1: this can only be the set $S = \{18\}$.
2	9	8: this can be any set $S = \{a, 9\}$, where $1 \leq a \leq 8$.
3	6	$\binom{5}{2} = 10$: the greatest element must be 6, and there are $\binom{5}{2}$ ways of selecting the two other elements of S.
6	3	0: this is impossible because S cannot have 6 elements, the greatest of which is 3.
9	2	0: this is impossible because S cannot have 9 elements, the greatest of which is 2.
18	1	0: this is impossible because S cannot have 18 elements, the greatest of which is 1.

Thus the answer is $1 + 8 + 10 = \mathbf{19}$.

Problem 2. A class of 218 students takes a test. Each student's score is an integer from 0 to 100, inclusive. Compute the greatest possible difference between the mean and the median scores.

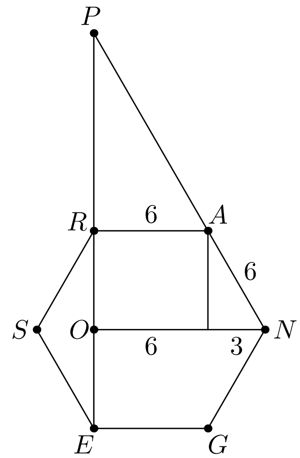
Solution 2. Intuitively, the maximum difference can be attained by forcing the median to be 0 by selecting just enough 0 scores, and then maximizing the mean by maximizing the remaining scores. Such a distribution would have 110 scores of 0 and 108 scores of 100, giving a median score of 0 and a mean score of $\frac{108 \cdot 100}{218} = \frac{5400}{109}$. This gives a difference of $\frac{5400}{109}$.

To prove that this is maximal, assume that the median score is m. Then, by definition, at least 109 scores must be at most m. Further, the remaining 109 scores are at most 100, so the mean μ satisfies $\mu \leq \frac{1}{218}(109m + 10900) = \frac{m+100}{2}$. Therefore $\mu - m \leq \frac{100-m}{2}$.

If $m = 0$, then the maximum mean is found as above. If $m \geq 1$, then $\mu - m \leq \frac{100-1}{2} = 49.5 < \frac{5400}{109}$, and the difference is greater than it is in the above case with $m = 0$. There only remains the possibility that $m = \frac{1}{2}$. For this to be the case, 109 scores must be 0 and one score must be 1. To maximize the mean, the remaining 108 scores must be 100. However, compared to the case of having 110 scores of 0 and 108 of 100, this increases the median by $\frac{1}{2}$ but increases the mean by only 1218, so the difference will certainly be smaller. Thus the answer is $\mathbf{\frac{5400}{109}}$.

Problem 3. Regular hexagon $RANGES$ has side length 6. Pentagon $RANGE$ is revolved 360° about the line containing $\overline{RE}$ to obtain a solid. The volume of the solid is $k \cdot \pi$. Compute k.

Solution 3. First note that because $\overline{RA}$ and $\overline{EG}$ are parallel, the solid obtained will consist of two conic frustums, each with a base along the plane through $\overline{NS}$ and perpendicular to $\overline{ER}$. It is easier to compute the volume of one of these conic frustums at a time. Let O be the midpoint of $\overline{ER}$, and let $\overrightarrow{NA}$ and $\overrightarrow{ER}$ intersect at P. Then the conic frustum is a cone with radius ON and height OP, cut off by the plane containing $\overline{AR}$ and perpendicular to $\overline{ER}$.



Note that $ON = NS - OS = 12 - 3 = 9$, and $OP = OR + RP = 3\sqrt{3} + 6\sqrt{3} = 9\sqrt{3}$. Then the volume of one of the conic frustums is

$$\frac{1}{3}\pi\left(9^2\right)\left(9\sqrt{3}\right) - \frac{1}{3}\pi\left(6^2\right)\left(6\sqrt{3}\right) = 171\sqrt{3}\,\pi.$$

The volume of the entire solid is therefore $342\sqrt{3}\cdot\pi$, so $k = \mathbf{342\sqrt{3}}$.

Problem 4. A fair 12-sided die has faces numbered 1 through 12. The die is rolled twice, and the results of the two rolls are x and y, respectively. Given that $\tan(2\theta) = \frac{x}{y}$ for some θ with $0 < \theta < \frac{\pi}{2}$, compute the probability that $\tan\theta$ is rational.

Solution 4. Suppose $\tan 2\theta = \frac{x}{y}$. Then the identity for the tangent of a sum gives $\frac{2\tan\theta}{1-\tan^2\theta} = \frac{x}{y}$, which is equivalent to $x - x\tan^2\theta = 2y\tan\theta$. Rearranging gives a quadratic equation in $\tan\theta$: $x\tan^2\theta + 2y\tan\theta - x = 0$. Thus $\tan\theta$ is rational if and only if the discriminant is a perfect square. The discriminant is $(2y)^2 - 4\cdot x\cdot(-x) = 4(x^2+y^2)$, which is a perfect square if and only if $x^2 + y^2$ is a perfect square. The only Pythagorean triples (a, b, c) with $a, b \leq 12$ are $(3, 4, 5), (6, 8, 10), (9, 12, 15)$, and $(5, 12, 13)$. Therefore the only (x, y) that give a rational value for $\tan\theta$ are $(3, 4), (4, 3), (6, 8), (8, 6), (9, 12), (12, 9), (5, 12), (12, 5)$. The desired probability is therefore $\frac{8}{12\cdot 12} = \mathbf{\frac{1}{18}}$.

Problem 5. The absolute values of the five complex roots of $z^5 - 5z^2 + 53 = 0$ all lie between the positive integers a and b, where $a < b$ and $b - a$ is minimal. Compute the ordered pair (a, b).

Solution 5. First note that the question asks for the tightest possible integer bounds on the magnitude of a zero of the complex polynomial $z^5 - 5z^2 + 53$, essentially describing the smallest complex annulus with integer radii that contains the roots.

Now recall two important properties of the absolute value function: $|ab| = |a||b|$ and $|a + b| \leq |a| + |b|$ for all complex numbers a and b. The latter also implies that $|a| - |b| \leq |a - b|$. In order to have $z^5 - 5z^2 + 53 = 0$, it must be the case that $|z^5 - 5z^2| = 53$. This is enough to restrict the possible magnitude of z.

Suppose that $|z| > 3$. Then $|z^5 - 5z^2| = |z|^2|z^3 - 5| > 9|z^3 - 5| \geq 9(|z|^3 - 5) > 9(27 - 5) = 198 > 53$. So $|z| \leq 3$.

Suppose that $|z| < 2$. Then $|z^5 - 5z^2| = |z|^2|z^3 - 5| < 4|z^3 - 5| \leq 4(|z|^3 + |5|) < 4(8 + 5) = 51 < 53$. So $|z| \geq 2$.

Therefore $2 \leq |z| \leq 3$ for all roots z of $z^5 - 5z^2 + 53$. Because $a < b$, and a and b are integers, $b - a$ is at least 1, so this is the best possible bound. The answer is $\mathbf{(2, 3)}$.

Problem 6. Let S be the set of points (x, y) whose coordinates satisfy the system of equations:

$$\lfloor x \rfloor \cdot \lceil y \rceil = 20$$
$$\lceil x \rceil \cdot \lfloor y \rfloor = 18.$$

Compute the least upper bound of the set of distances between points in S.

Solution 6. First note that x and y cannot both be integers because otherwise, the given system would be inconsistent. Now suppose that exactly one of x or y is an integer. If x is an integer and $\lfloor y \rfloor = b$, then $\lceil y \rceil = b+1$, and multiplying the two given equations and simplifying results in the equation $x^2 \cdot b(b+1) = 360$ (†). Because $1 \leq x^2 < 360$, the only possible values of x^2 are the perfect square factors of 360, namely 1, 4, 9, and 36. Substituting in $x^2 = 1, 9$, and 36 into (†) results in non-integral solutions for b. But substituting $x^2 = 4$ into (†) results in $b^2 + b - 90 = (b+10)(b-9) = 0$, and $b = -10$ or $b = 9$. If $x = 2$, then $\lfloor y \rfloor = 9$ and $\lceil y \rceil = 10$, and this yields the solutions $(2, y)$, where $9 < y < 10$. On the other hand, if $x = -2$, then $\lfloor y \rfloor = -9$ and $\lceil y \rceil = -10$, which is impossible. By a similar argument, if x is not an integer and y is an integer, this results in the solutions $(x, -2)$, where $-10 < x < -9$. Thus if $\mathcal{I}_1$ is the open interval $(9, 10)$ and $\mathcal{I}_2$ is the open interval $(-10, -9)$, then the solutions (x, y) to the given system where exactly one of x and y is an integer are the ordered pairs belonging to the union of the two sets $\{2\} \times \mathcal{I}_1$* and $\mathcal{I}_2 \times \{-2\}$. Graphically, these represent segments of unit length (one vertical, one horizontal) that do not include the endpoints. For later reference, let $\mathcal{I}_x = \mathcal{I}_2 \times \{-2\}$ and let $\mathcal{I}_y = \{2\} \times \mathcal{I}_1$.

Now suppose that neither x nor y is an integer and that $\lfloor x \rfloor = a$ and $\lfloor y \rfloor = b$. Then $\lceil x \rceil = a+1, \lceil y \rceil = b+1$, and multiplying the two given equations and simplifying results in the equation $a(a+1)b(b+1) = 360$ (‡). By noting the factorization $360 = 3 \cdot 4 \cdot 5 \cdot 6$, the following ordered pairs (a, b) satisfy (‡):

$$(3,5); \quad (-4,5); \quad (-4,-6); \quad (3,-6); \quad (5,3); \quad (5,-4); \quad (-6,-4); \quad (-6,3).$$

However, only $(a, b) = (5, 3)$ and $(-4, -6)$ satisfy the given system:

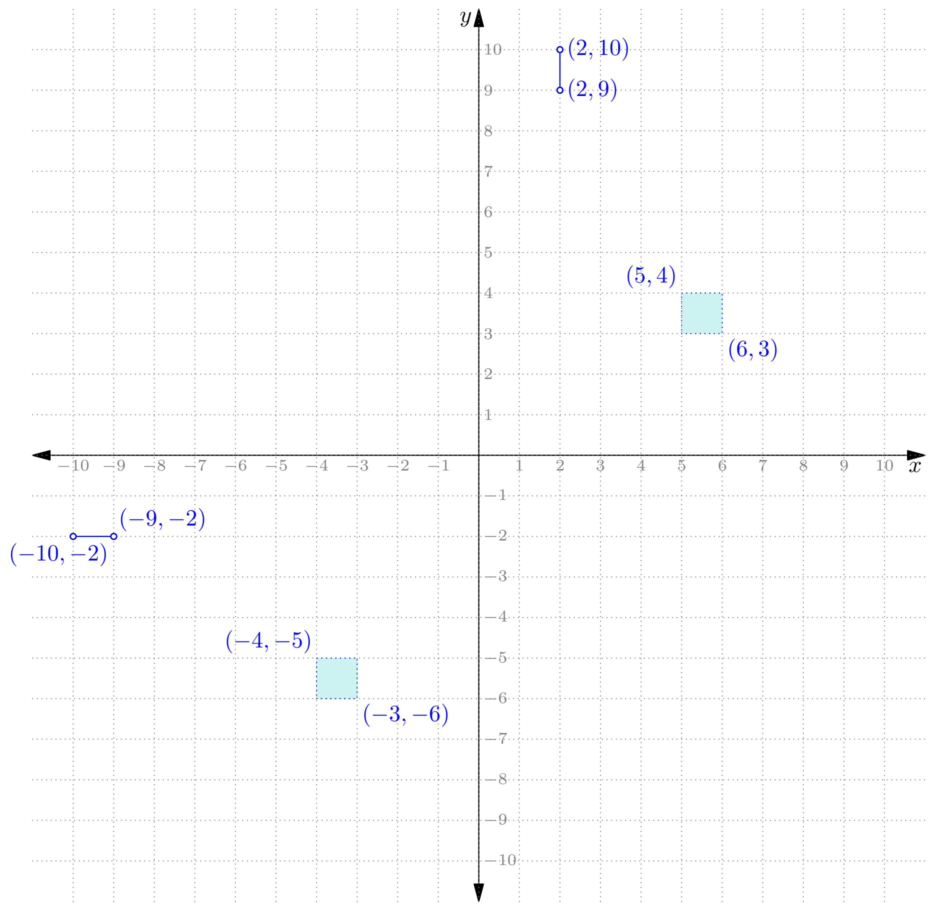
$$a(b+1) = 20$$
$$(a+1)b = 18.$$

Let $\mathcal{I}_3$ and $\mathcal{I}_4$ denote the open intervals $(5, 6)$ and $(3, 4)$, respectively and let $\mathcal{I}_5$ and $\mathcal{I}_6$ denote the open intervals $(-4, -3)$ and $(-6, -5)$, respectively. Then the solutions (x, y) to the given system in which neither x nor y is an integer are given by the union of the two regions $\mathcal{R}_1$ and $\mathcal{R}_2$, defined by

$$\mathcal{R}_1 = \mathcal{I}_3 \times \mathcal{I}_4 \quad \text{and} \quad \mathcal{R}_2 = \mathcal{I}_5 \times \mathcal{I}_6.$$

Each of $\mathcal{R}_1$ and $\mathcal{R}_2$ is the interior of a unit square ($\mathcal{R}_1$ lies in the first quadrant and $\mathcal{R}_2$ lies in the third quadrant). Also note that the solutions are symmetric about the line $y = -x$. A plot of the points of S is shown.

*Here, $\times$ denotes the Cartesian product. I.e., if P and Q are two sets, then $P \times Q = \{(p, q) \mid p \in P \text{ and } q \in Q\}$.



The answer to the problem is $\max\{\ell_1, \ell_2, \ell_3\}$, where:

- ℓ_1 is the least upper bound of the distances between a point in $\mathcal{I}_x$ and a point in $\mathcal{I}_y$;
- ℓ_2 is the least upper bound of the distances between a point in $\mathcal{R}_1$ and a point in $\mathcal{R}_2$;
- ℓ_3 is the least upper bound of the distances between a point in $\mathcal{I}_x \cup \mathcal{I}_y$ and a point in $\mathcal{R}_1 \cup \mathcal{R}_2$.

Compute ℓ_1 by taking the distance between the extremal boundary points of $\mathcal{I}_x$ and $\mathcal{I}_y$, namely $(-10, -2)$ and $(2, 10)$, respectively. This distance is $\sqrt{2 \cdot 12^2} = \sqrt{288}$.

Compute ℓ_2 by taking the distance between the extremal boundary points of $\mathcal{R}_1$ and $\mathcal{R}_2$, namely $(6, 4)$ and $(-4, -6)$, respectively. This distance is $\sqrt{2 \cdot 10^2} = \sqrt{200}$.

Compute ℓ_3 by taking the distance between the extremal boundary points of $\mathcal{I}_x$ and $\mathcal{R}_1$ (or of $\mathcal{I}_y$ and $\mathcal{R}_2$, owing to the symmetry), namely $(-10, -2)$ and $(6, 4)$, respectively. This distance is $\sqrt{16^2 + 6^2} = \sqrt{292}$.

Thus the answer is $\max\{\sqrt{288}, \sqrt{200}, \sqrt{292}\} = \sqrt{292} = \mathbf{2\sqrt{73}}$.

Problem 7. Compute the least integer $d > 0$ for which there exist distinct lattice points A, B, and C in the coordinate plane, each exactly $\sqrt{d}$ units from the origin, satisfying $\csc(\angle ABC) > 2018$.

Solution 7. The points A, B, and C lie on a circle with diameter $2\sqrt{d}$. Consequently, by the Extended Law of Sines,

$$2\sqrt{d} = \frac{AC}{\sin(\angle ABC)} \;\rightarrow\; \csc(\angle ABC) = \frac{2\sqrt{d}}{AC}\,.$$

The solution now proceeds in three cases.

- It is impossible that $AC = 1$. Indeed, if $A = (x, y)$, then without loss of generality, assume $C = (x+1, y)$. Then $d = x^2 + y^2 = (x+1)^2 + y^2$, which implies $2x + 1 = 0$, contradiction.
- Suppose $AC = \sqrt{2}$. If $A = (x, y)$, then without loss of generality, assume $C = (x+1, y+1)$, by reflecting or rotating as necessary. Then $d = x^2 + y^2 = (x+1)^2 + (y+1)^2$, hence $2x + 1 + 2y + 1 = 0$. It follows that $y = -(x+1)$ and d must be of the form $d = x^2 + (x+1)^2$. In that case,

$$\csc(\angle ABC) = \frac{2\sqrt{d}}{\sqrt{2}} = \sqrt{2(x^2 + (x+1)^2)}$$

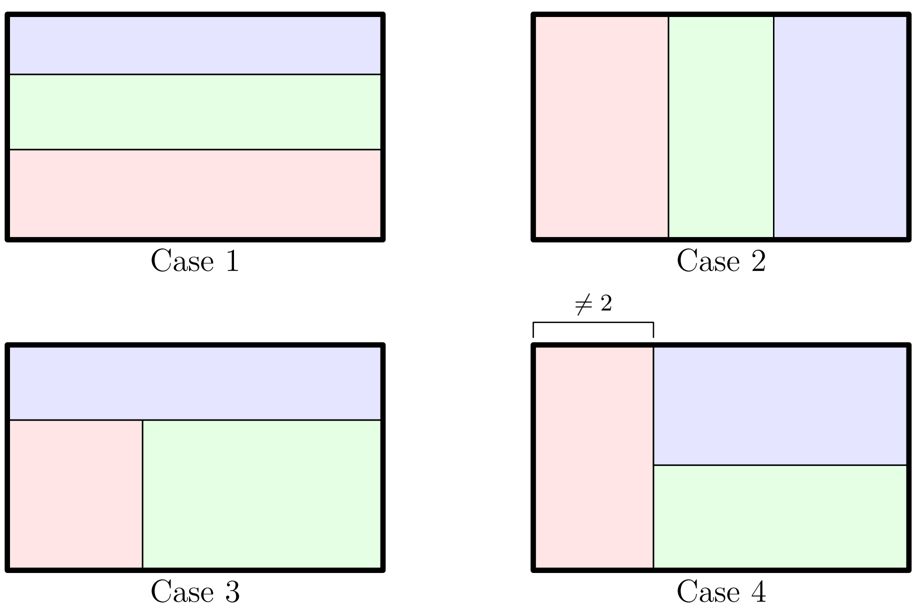
  and the least x for which this exceeds 2018 is $x = 1009$, meaning $d = 1009^2 + 1010^2$.
- Finally assume $AC \geq \sqrt{3}$. Then $2018 < \frac{2\sqrt{d}}{AC} \leq \sqrt{\frac{4}{3}d}$, which would mean $d > 2018^2 \cdot \frac{3}{4} > 1009^2 + 1010^2$. Thus all values of d achieved in this case are greater than the value of d in the second case.

In conclusion, the least possible value of d is $1009^2 + 1010^2 = \mathbf{2038181}$. This value of d is indeed possible, as the points $A = (1009, 1010)$, $B = (-1009, -1010)$, and $C = (1010, 1009)$ satisfy the conditions of the problem.

Problem 8. Compute the number of unordered collections of three integer-area rectangles such that the three rectangles can be assembled without overlap to form one 3×5 rectangle. (For example, one such collection contains one 3×3 and two 1×3 rectangles, and another such collection contains one 3×3 and two 2×1.5 rectangles. The latter collection is equivalent to the collection of two 1.5×2 rectangles and one 3×3 rectangle.)

Solution 8. In order to enumerate the possible rectangles it is necessary to consider several cases. One possible approach is presented below.

Consider the possible dissections of the given 3×5 rectangle. There are four shapes which can be achieved, as shown below.



Begin by counting the number of unordered collections that arise from each case.

- In Case 1, a collection corresponds to a multiset $\{a, b, c\}$ satisfying $a + b + c = 15$ (as the three rectangles can be reordered freely). Assume, without loss of generality, that $a \leq b \leq c$. To eliminate the inequality constraints, let $x = a - 1$, $y = b - a$, $z = c - b$ denote nonnegative integers. Accordingly, $a = x + 1$, $b = x + y + 1$, $z = x + y + z + 1$, and the given equation now reads $x + 2y + 3z = 12$ or, equivalently, $2y + 3z \le 12$. The number of solutions in nonnegative integers is then

$$\sum_{z=0}^{4} \left(1 + \left\lfloor \frac{12 - 3z}{2} \right\rfloor\right) = 7 + 5 + 4 + 2 + 1 = 19.$$

- Case 2 has 19 solutions by repeating the argument from Case 1 verbatim.
- In Case 3, denote by $15 - n$ the area of the top rectangle. The number of possible dissections of the lower rectangle is then exactly $\lfloor n/2 \rfloor$. Moreover, the top rectangle (because it has a side length of 5) is never congruent to either of the lower rectangles. Thus the number of solutions in this case is

$$\sum_{n=1}^{14} \left\lfloor \frac{n}{2} \right\rfloor = 49.$$

- In Case 4, assume the leftmost region does not have width 2. This ensures that Case 2 and Case 4 will be disjoint from each other, because the two rectangles on the right will not form a 3×3 square that could be rotated to obtain a dissection already counted in Case 2.

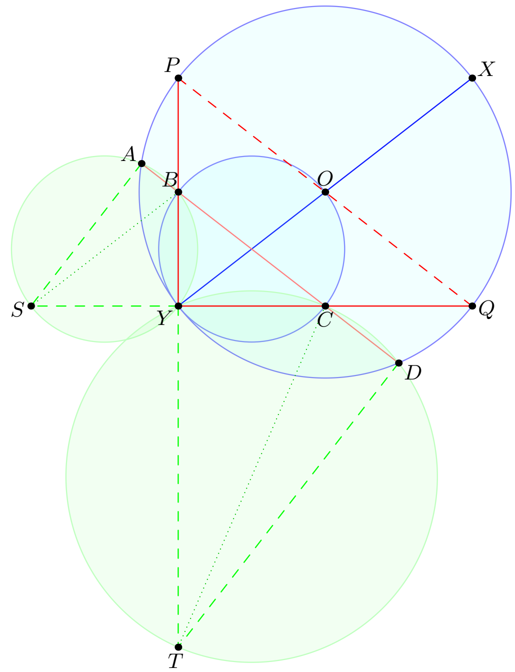
  In order to count the number of such configurations, denote by $15 - n$ the area of the left rectangle, where $15 - n \neq 3 \cdot 2$ or, equivalently, $n \neq 9$. Then repeating the logic of Case 3 gives a count of

$$\sum_{\substack{n=1 \\ n \neq 9}}^{14} \left\lfloor \frac{n}{2} \right\rfloor = 49 - \left\lfloor \frac{9}{2} \right\rfloor = 45.$$

It has already been checked that Cases 3 and 4 are disjoint, and the other pairs of cases are seen to be mutually exclusive by comparing the number of sides of length 5 in the dissection. Consequently, the final answer is $19 + 19 + 49 + 45 = \mathbf{132}$.

Problem 9. Let Γ be a circle with diameter $\overline{XY}$ and center O, and let γ be a circle with diameter $\overline{OY}$. Circle ω_1 passes through Y and intersects Γ and γ again at A and B, respectively. Circle ω_2 also passes through Y and intersects Γ and γ again at D and C, respectively. Given that $AB = 1$, $BC = 4$, $CD = 2$, and $AD = 7$, compute the sum of the areas of ω_1 and ω_2.

Solution 9. Note that A, B, C, and D are collinear, in that order, because $AD = AB + BC + CD$. Extend $\overline{YB}$ and $\overline{YC}$ to meet Γ again at P and Q, respectively. Then $\overline{BC}$ is the midline of $\triangle YPQ$, by homothety. By power of a point, $AB \cdot BD = 1 \cdot (4+2) = PB \cdot BY = BY^2$, so $BY = \sqrt{6}$. Similarly, $CY = \sqrt{2 \cdot (4+1)} = \sqrt{10}$. In particular, $\triangle BYC$ is right.



Now let S and T be the antipodes of B and C on ω_1 and ω_2, respectively. It remains to evaluate BS and CT. First note that $m\angle BYS = 90^\circ$ and $m\angle CAS = m\angle BAS = 90^\circ$. Because $m\angle BYC = 90^\circ$, it follows that points C, Y, and S are collinear and that $\triangle CYB \sim \triangle CAS$. Therefore

$$AS = \frac{YB}{YC} \cdot AC = \sqrt{\frac{6}{10}} \cdot 5 = \sqrt{15},$$

hence $BS = \sqrt{AS^2 + AB^2} = 4$. In the same fashion, with $\triangle BYC \sim \triangle BDT$,

$$DT = \frac{YC}{BY} \cdot BD = \sqrt{\frac{10}{6}} \cdot 6 = \sqrt{60},$$

hence $CT = \sqrt{DT^2 + CD^2} = 8$. So the sum of the areas of ω_1 and ω_2 is $\pi(2^2 + 4^2) = \mathbf{20\pi}$.

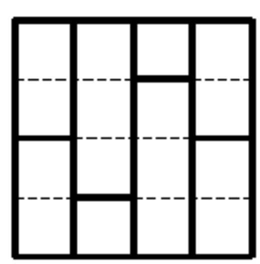
Problem 10. In the number puzzle below, clues are given for the four rows, each of which contains a four-digit number. Cells inside a region bounded by bold lines must all contain the same digit, and each of the eight regions contains a different digit. The variables in the clues are all positive integers. Complete the number puzzle.

1: $4^a + 13^b + 14^c$

2: $5^p + 13^q + 17^r$

3: $4^x + 5^y + 31^z$

4: the average of the other 3 rows



Solution 10. The following solution only relies on the clues for the third and fourth rows. Label the digits in the array as follows.

A	B	C	D
A	B	Y	D
W	B	Y	Z
W	X	Y	Z

The solution proceeds in three steps.

Step 1. The final row is the average of the first three rows, which implies

$$3(1000W + 100X + 10Y + Z) = 1000(2A + W) + 100(3B) + 10(C + 2Y) + (2D + Z).$$

This rearranges to

$$0 = 2000(A - W) + 300(B - X) + 10(C - Y) + 2(D - Z).$$

By comparing the magnitudes of $|2000(A - W)|$ to $|300(B - X) + 10(C - Y) + 2(D - Z)|$, it is impossible to have $|A - W| \geq 2$, hence $A - W = \pm 1$. A similar argument gives $B - X = \mp 7$, and then $C - Y = \pm 9$, $D - Z = \pm 5$, where the signs correspond. In particular, $\{C, Y\} = \{0, 9\}$.

Step 2. Next, the number in the third row satisfies

$$4^x + 5^y + 31^z \equiv 4^x + 6 \pmod{10}$$

and consequently either $Z = 0$ or $Z = 2$. However, one of C and Y must be 0, hence $\boxed{Z = 2}$. This can only occur if $\boxed{D = 7}$, which determines all the signs in the previous step: it follows that $A - W = -1$, $B - X = -7$, $C - Y = 9$, $D - Z = 5$. In particular, $\boxed{C = 9}$ and $\boxed{Y = 0}$.

Step 3. Because $B - X = -7$ and the digits 0 and 2 have already been used, it follows that $\boxed{B = 1}$ and $\boxed{X = 8}$. The third clue now reads

$$4^x + 5^y + 31^z = \underline{W}\ \underline{1}\ \underline{0}\ \underline{2}.$$

The mod 10 calculation in the previous step implies that x is even and in particular, $x > 1$. Taking the previous equation modulo 8 gives $5^y + 31^z \equiv 102 \equiv 6 \pmod{8}$, which can only occur if y is odd and z is even. This implies $z = 2$ (as $z \leq 2$). Therefore $4^x + 5^y \equiv 141 \pmod{1000}$ and because $x \leq 6$ is even, $y \leq 5$, and this forces $x = 2$ and $y \in \{3, 5\}$. The value $y = 3$ would give $W = 1$, which is not permitted because $B = 1$, so $y = 5$ and $\boxed{W = 4}$. Finally, $\boxed{A = 5}$.

The completed number puzzle reads as follows.

5	1	9	7
5	1	0	7
4	1	0	2
4	8	0	2