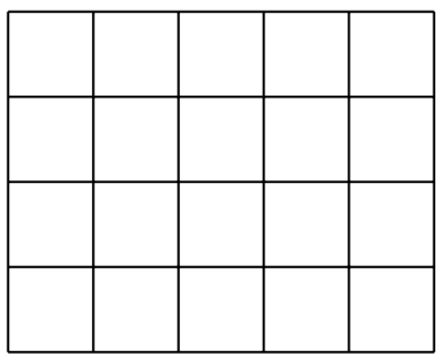
CT ARML Team, 2026
Team Selection Test 2

1. In triangle ABC, $AB = 6$, $AC = 4$, $BC = 5$. The angle bisector of $\angle A$ intersects $\overline{BC}$ at D. Compute $(AD)^2$.

2. A computer is programmed to choose an integer between 1 and 99, inclusive, so that the probability that it selects any integer x (with $1 \le x \le 99$) is equal to $\log_{100}\left(1 + \frac{1}{x}\right)$. Suppose that the probability that $81 \le x \le 99$ is equal to 2 times the probability that $x = n$ for some integer n. What is the value of n?

3. Triangle ABC is isosceles with $AB = AC = 1$ and $\angle A = 30^\circ$. If $\triangle BCD$ is an isosceles right triangle with $\angle C = 90^\circ$ and does not overlap $\triangle ABC$, then $AD^2 = p + (-1)^n\sqrt{q}$, where p and q are positive integers and n is 0 or 1. Compute $p + q + n$.

4. Consider a rectangular grid of squares with four rows and five columns, as shown in the diagram below. Compute the number of ways to place a mark in one square in each column so that the marked squares in neighboring columns are always in the same or neighboring rows. For example, if the third column has its mark in the second row, then both the second and fourth columns must have their marks in the first, second, or third rows.



5. Compute the smallest positive integer that has exactly 26 positive composite factors. That is, factors other than 1 or a prime number.

6. The numbers $1, 2, \ldots, 13$ are written down, one at a time, in a random order. The probability that at no time during this process the sum of all written numbers is divisible by 3 is $\frac{p}{q}$, where p and q are relatively prime positive integers. Compute $p + q$.

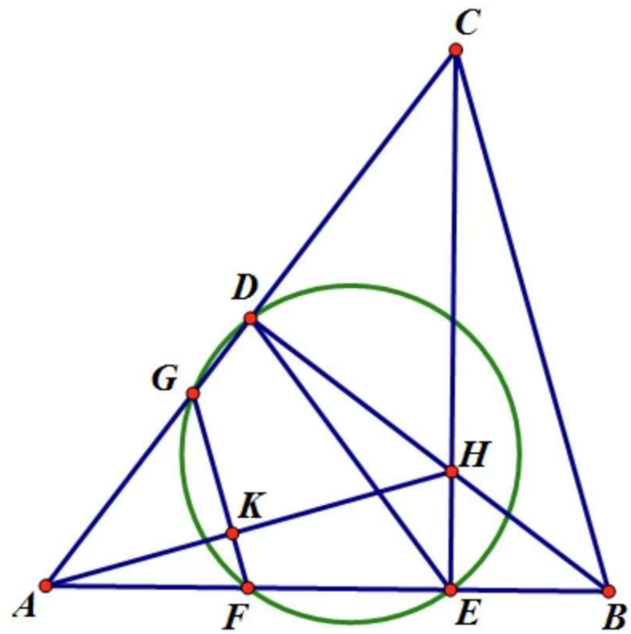
7. Compute the smallest positive integer k such that

$$\left(\frac{\sin 20^\circ}{\cos 25^\circ} + \frac{\sin 25^\circ}{\cos 20^\circ} i\right)^k \in \mathbb{R}.$$

8. Compute the largest positive integer that divides $p^2 - 1$ for all prime numbers $p > 3$.

9. Suppose that the numbers $\log_3(6x)$, $\log_9(18x)$, $\log_{27}(12x)$ are in arithmetic progression. Then $x = \frac{p}{q}$, where p and q are relatively prime positive integers. Compute $p + q$.

10. In the acute triangle ABC, as shown in the diagram below, the altitude $\overline{CE}$ from C to $\overline{AB}$ and the altitude $\overline{BD}$ from B to $\overline{AC}$ intersect at point H. The circle with $\overline{DE}$ as its diameter intersects $\overline{AB}$ and $\overline{AC}$ again at points F and G, respectively. Line $\overline{FG}$ intersects $\overline{AH}$ at point K. Given that $BC = 25$, $BD = 20$, and $BE = 7$, the length AK is $\frac{p}{q}$, where p and q are relatively prime positive integers. Compute $p + q$.



CT ARML Team, 2026
Team Selection Test 2
Answers

1. 18
2. 9
3. 8
4. 178
5. 720
6. 183
7. 18
8. 24
9. 13
10. 241

CT ARML Team, 2026
Team Selection Test 2
Solutions

1. In triangle ABC, $AB = 6$, $AC = 4$, $BC = 5$. The angle bisector of $\angle A$ intersects $\overline{BC}$ at D. Compute $(AD)^2$.
[Answer: 18]

Solution (Sumanyu Nandecha)
By the Angle Bisector Theorem,

$$\frac{BD}{DC} = \frac{AB}{AC} = \frac{6}{4} = \frac{3}{2}.$$

Since $BC = 5$, we get $BD = 3$ and $DC = 2$. Applying Stewart's Theorem to cevian AD:

$$AB^2 \cdot DC + AC^2 \cdot BD = BC(AD^2 + BD \cdot DC),$$

$$6^2(2) + 4^2(3) = 5(AD^2 + 6) \implies 90 = 5AD^2 + 30 \implies AD^2 = \boxed{18}.$$

2. A computer is programmed to choose an integer between 1 and 99, inclusive, so that the probability that it selects any integer x (with $1 \le x \le 99$) is equal to $\log_{100}\left(1 + \frac{1}{x}\right)$. Suppose that the probability that $81 \le x \le 99$ is equal to 2 times the probability that $x = n$ for some integer n. What is the value of n?
[Answer: 9]

Solution (Sumanyu Nandecha)
Using log properties to telescope the sum,

$$P(81 \le x \le 99) = \sum_{k=81}^{99} \log_{100}\left(1 + \frac{1}{k}\right) = \log_{100} \prod_{k=81}^{99} \frac{k+1}{k} = \log_{100} \frac{100}{81}.$$

Setting this equal to $2P(x = n) = \log_{100}\left(1 + \frac{1}{n}\right)^2$, we get

$$\left(1 + \frac{1}{n}\right)^2 = \frac{100}{81} \implies 1 + \frac{1}{n} = \frac{10}{9} \implies n = \boxed{9}.$$

3. Triangle ABC is isosceles with $AB = AC = 1$ and $\angle A = 30^\circ$. If $\triangle BCD$ is an isosceles right triangle with $\angle C = 90^\circ$ and does not overlap $\triangle ABC$, then $AD^2 = p + (-1)^n\sqrt{q}$, where p and q are positive integers and n is 0 or 1. Compute $p + q + n$.
[Answer: 8]

Solution (Abby Kesmodel)
By the Law of Cosines,

$$BC^2 = 1^2 + 1^2 - 2(1)(1)\cos 30^\circ = 2 - \sqrt{3},$$

so $BC = \sqrt{2 - \sqrt{3}} = \sqrt{\frac{(\sqrt{3}-1)^2}{2}} = \frac{\sqrt{6}-\sqrt{2}}{2}$.

Let x be the height from A to BC. By the Pythagorean Theorem, $\left(\frac{BC}{2}\right)^2 + x^2 = 1$, so

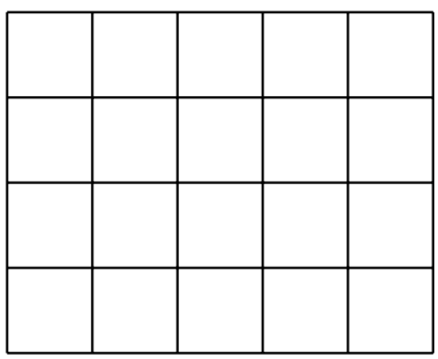
$$x^2 = \frac{8+2\sqrt{12}}{16} = \left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)^2 \implies x = \frac{\sqrt{6}+\sqrt{2}}{4}.$$

Since $\triangle BCD$ is an isosceles right triangle with right angle at C, $CD = BC = \frac{\sqrt{6}-\sqrt{2}}{2}$, and D lies on the opposite side of BC from A. Dropping the altitude from A to line BC and applying the Pythagorean Theorem:

$$\begin{aligned} AD^2 &= (x+CD)^2 + \left(\tfrac{BC}{2}\right)^2 \\ &= \left(\tfrac{\sqrt{6}+\sqrt{2}}{4} + \tfrac{\sqrt{6}-\sqrt{2}}{2}\right)^2 + \left(\tfrac{\sqrt{6}-\sqrt{2}}{4}\right)^2 \\ &= \left(\tfrac{3\sqrt{6}-\sqrt{2}}{4}\right)^2 + \left(\tfrac{\sqrt{6}-\sqrt{2}}{4}\right)^2 \\ &= \tfrac{56-6\sqrt{12}}{16} + \tfrac{8-2\sqrt{12}}{16} = \tfrac{64-16\sqrt{3}}{16} = 4-\sqrt{3}. \end{aligned}$$

Thus $p = 4$, $q = 3$, $n = 1$, and $p + q + n = \boxed{8}$.

4. Consider a rectangular grid of squares with four rows and five columns, as shown in the diagram below. Compute the number of ways to place a mark in one square in each column so that the marked squares in neighboring columns are always in the same or neighboring rows. For example, if the third column has its mark in the second row, then both the second and fourth columns must have their marks in the first, second, or third rows.



[Answer: 178]

Solution (Abby Kesmodel)

Reframe the problem as counting paths from column 1 to column 5, where each step moves into the next column either straight across, up-right, or down-right (no vertical moves within a column). The number of paths reaching cell (a, b) is the sum of the paths reaching $(a-1, b-1)$, $(a-1, b)$, and $(a-1, b+1)$.

Starting with 1 in each cell of column 1, fill in the table:

1	2	5	13	34
1	3	8	21	55
1	3	8	21	55
1	2	5	13	34

Summing the entries in column 5: $34 + 55 + 55 + 34 = \boxed{178}$.
Aside: Notice the Fibonacci pattern. With $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, the entries in column n are $F_{2n-2}, F_{2n-1}, F_{2n-1}, F_{2n-2}$, and the column sum is $2F_{2n}$.

5. Compute the smallest positive integer that has exactly 26 positive composite factors. That is, factors other than 1 or a prime number.
[Answer: 720]

Solution (Girish Prasad)
Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, so n has $(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_k+1)$ total factors. If n has 26 composite factors, k prime factors, and the factor 1, then n has $k + 27$ total factors.
We inspect values of k:

- $k = 1$: n has 28 factors, so $n = p_1^{27}$. Smallest: 2^{27}.
- $k = 2$: n has 29 factors. Since 29 is prime and each $\alpha_i + 1 \geq 2$, no solutions.
- $k = 3$: n has $30 = 5 \cdot 3 \cdot 2$ factors. Smallest: $2^4 \cdot 3^2 \cdot 5 = 720$.
- $k = 4$: n has 31 factors. Since 31 is prime, no solutions.
- $k \geq 5$: Minimum $n \geq 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310 > 720$.

The answer is $\min(2^{27}, 720) = \boxed{720}$.

6. The numbers $1, 2, \ldots, 13$ are written down, one at a time, in a random order. The probability that at no time during this process the sum of all written numbers is divisible by 3 is $\frac{p}{q}$, where p and q are relatively prime positive integers. Compute $p + q$.
[Answer: 183]

Solution (Abby Kesmodel)
Among $1, 2, \ldots, 13$, there are 5 numbers $\equiv 1 \pmod 3$, 4 numbers $\equiv 2 \pmod 3$, and 4 numbers $\equiv 0 \pmod 3$. Zeros don't affect divisibility of the running sum by 3, so first determine the order of the nonzero residues.
If the first nonzero residue is 1, then to keep the running sum nonzero mod 3, we must follow it with another 1 (writing 2 would give sum 0). The next must be 2 (the sum is now $2 \pmod 3$). Continuing this way, the residue order must be $1, 1, 2, 1, 2, 1, 2, 1, 2$, using exactly 5 ones and 4 twos.
Starting with a 2 would require more 2's than 1's, which we don't have. So the order of nonzero residues is forced.
The four zeros can be placed in any positions *except* the very first. This gives $\binom{12}{4}$ ways. Within each residue class, the actual numbers can be arranged in any order, giving $5! \cdot 4! \cdot 4!$ orderings.

The probability is

$$\frac{\binom{12}{4} \cdot 5! \cdot 4! \cdot 4!}{13!} = \frac{1}{182},$$

so $p + q = 1 + 182 = \boxed{183}$.

7. Compute the smallest positive integer k such that

$$\left(\frac{\sin 20^\circ}{\cos 25^\circ} + \frac{\sin 25^\circ}{\cos 20^\circ} i \right)^k \in \mathbb{R}.$$

[Answer: 18]

Solution (Sumanyu Nandecha)
Combining over a common denominator:

$$z = \frac{\sin 20^\circ \cos 20^\circ + i \sin 25^\circ \cos 25^\circ}{\cos 20^\circ \cos 25^\circ}.$$

Using $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$:

$$z = \frac{\frac{1}{2}\sin 40^\circ + \frac{i}{2}\sin 50^\circ}{\cos 20^\circ \cos 25^\circ} = \frac{1}{2\cos 20^\circ \cos 25^\circ}(\sin 40^\circ + i \sin 50^\circ).$$

Since $\sin 40^\circ = \cos 50^\circ$,

$$z = \frac{1}{2\cos 20^\circ \cos 25^\circ}(\cos 50^\circ + i \sin 50^\circ) = r\, e^{i \cdot 50^\circ}.$$

Then $z^k \in \mathbb{R}$ iff $\sin(50^\circ k) = 0$, i.e., $50k = 180m$ for some integer m, or $k = \frac{18}{5}m$. The smallest positive integer k occurs at $m = 5$, giving $k = \boxed{18}$.

8. Compute the largest positive integer that divides $p^2 - 1$ for all prime numbers $p > 3$.
[Answer: 24]

Solution (Sumanyu Nandecha)
Factor $p^2 - 1 = (p-1)(p+1)$.
Divisibility by 8. Since $p > 3$ is odd, $p - 1$ and $p + 1$ are consecutive even integers, so one is divisible by 4. Therefore $8 \mid (p-1)(p+1)$.
Divisibility by 3. Since $p > 3$ is prime, $p \not\equiv 0 \pmod 3$, so $p^2 \equiv 1 \pmod 3$.
Since $\gcd(8, 3) = 1$, we get $24 \mid p^2 - 1$. For $p = 5$, $p^2 - 1 = 24$, so the answer is $\boxed{24}$.

9. Suppose that the numbers $\log_3(6x)$, $\log_9(18x)$, $\log_{27}(12x)$ are in arithmetic progression, in that order. Then $x = \frac{p}{q}$, where p and q are relatively prime positive integers. Compute $p + q$.
[Answer: 13]

Solution (Girish Prasad)
Converting to base 3, the three terms are $\log_3(6x)$, $\frac{1}{2}\log_3(18x)$, $\frac{1}{3}\log_3(12x)$. The arithmetic progression condition gives

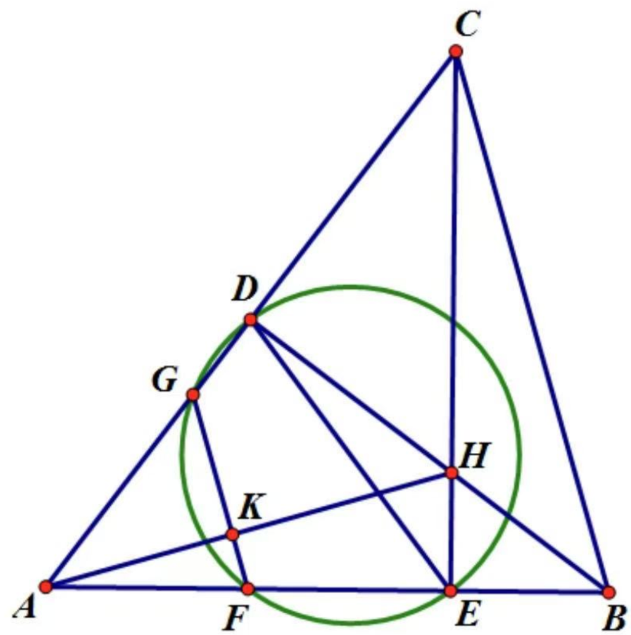
$$\log_3(6x) + \tfrac{1}{3}\log_3(12x) = \log_3(18x).$$

Multiplying through by 3 and rearranging:

$$\log_3(12x) = 3\log_3(18x) - 3\log_3(6x) = \log_3\left(\frac{18^3x^3}{6^3x^3}\right) = \log_3 27 = 3.$$

So $12x = 27$, giving $x = \frac{9}{4}$, and $p + q = \boxed{13}$.

10. In the acute triangle ABC, as shown in the diagram below, the altitude $\overline{CE}$ from C to $\overline{AB}$ and the altitude $\overline{BD}$ from B to $\overline{AC}$ intersect at point H. The circle with $\overline{DE}$ as its diameter intersects $\overline{AB}$ and $\overline{AC}$ again at points F and G, respectively. Line $\overline{FG}$ intersects $\overline{AH}$ at point K. Given that $BC = 25$, $BD = 20$, and $BE = 7$, the length AK is $\frac{p}{q}$, where p and q are relatively prime positive integers. Compute $p + q$.



[Answer: 241]

Solution (Abby Kesmodel; approach suggested by Vikram Sarkar)
By the Pythagorean Theorem, $CD = 15$ and $CE = 24$.
Since $\angle CDB = \angle BEC = 90^\circ$, quadrilateral $BCDE$ is cyclic with diameter BC. By Ptolemy's Theorem,

$$DE \cdot 25 + 15 \cdot 7 = 20 \cdot 24 \implies DE = 15.$$

So $\triangle CDE$ is isosceles with $CD = DE$, giving $\angle DCE = \angle DEC$. Combined with the inscribed-angle equalities $\angle DCE = \angle DBE$ and $\angle DEC = \angle DBC$ from cyclic $BCDE$, we get $\angle EBD = \angle CBD$. Hence $\triangle CDB \cong \triangle ADB$ by ASA, giving $AD = 15$, $AB = 25$, and $AE = AB - BE = 18$.
By SAS similarity, $\triangle ADE \sim \triangle ABC$ with ratio $\frac{AD}{AB} = \frac{15}{25} = \frac{3}{5}$ (equivalently, $DE = BC \cdot \frac{3}{5} = 15$). Now DE is the diameter of the new circle, just as BC was the diameter of the original; and F, G are the feet of the altitudes from D, E in $\triangle ADE$ (since $\angle DFE = \angle DGE = 90^\circ$). By the same reasoning, $\triangle AGF \sim \triangle ADE$ with ratio $\frac{3}{5}$, so $\triangle AGF \sim \triangle ABC$ with ratio $\left(\frac{3}{5}\right)^2 = \frac{9}{25}$.
Since K lies on $\overline{AH}$, which extends to the altitude from A to $\overline{BC}$, AK corresponds to this altitude in $\triangle ABC$. Using $[ABC] = \frac{1}{2} \cdot AC \cdot BD = \frac{1}{2} \cdot 30 \cdot 20 = 300$, the altitude from A has

length $\frac{2 \cdot 300}{25} = 24$. Therefore

$$AK = 24 \cdot \left(\frac{3}{5}\right)^2 = \frac{216}{25},$$

and $p + q = 216 + 25 = \boxed{241}$.

4 Power Question 2018: Partitions

Instructions: The power question is worth 50 points; each part's point value is given in brackets next to the part. To receive full credit, the presentation must be legible, orderly, clear, and concise. If a problem says "list" or "compute," you need not justify your answer. If a problem says "determine," "find," or "show," then you must show your work or explain your reasoning to receive full credit, although such explanations do not have to be lengthy. If a problem says "justify" or "prove," then you must prove your answer rigorously. Even if not proved, earlier numbered items may be used in solutions to later numbered items, but not vice versa. Pages submitted for credit should be NUMBERED IN CONSECUTIVE ORDER AT THE TOP OF EACH PAGE in what your team considers to be proper sequential order. PLEASE WRITE ON ONLY ONE SIDE OF THE ANSWER PAPERS. Put the TEAM NUMBER (not the team name) on the cover sheet used as the first page of the papers submitted. Do not identify the team in any other way.

Binary Partitions

A *binary partition* of a positive integer n is an ordered n-tuple of non-increasing integers, each of which is either 0 or a power of 2, whose sum is n. Each of the integers in the n-tuple is called a *part* of the partition. Each binary partition of n has n parts. Let $p_2(n)$ denote the number of binary partitions of n. For example, $p_2(3) = 2$ because of the two ordered triples $(2,1,0)$ and $(1,1,1)$.

1. Compute $p_2(n)$ for $n = 4$, 5, 6, and 7. [4 pts]

2. **a.** Show that $p_2(n) \leq p_2(n+1)$ for all positive integers n. [3 pts]

b. Is the inequality strict for sufficiently large n? Justify your answer. [3 pts]

3. **a.** Prove that if n is even and $n \geq 4$, then $p_2(n) = p_2\left(\frac{n}{2}\right) + p_2(n-2)$. [2 pts]

b. Find the least $n > 1$ such that $p_2(n)$ is odd, or prove that no such n exists. [2 pts]

Partial Orderings

A *partial ordering* on a set S is a relation, usually denoted $\preceq$, such that all of the following conditions are true:

- $a \preceq a$ for all $a \in S$ **(reflexivity property)**,
- $a \preceq b$ and $b \preceq c$ implies $a \preceq c$ for all $a, b, c \in S$ **(transitivity property)**, and
- $a \preceq b$ and $b \preceq a$ implies $a = b$ for all $a, b \in S$ **(antisymmetry property)**.

The word "partial" refers to the possibility that some two elements, a and b, may be *incomparable*, i.e., neither $a \preceq b$ nor $b \preceq a$ (these negative relations on $\preceq$ are sometimes written as $a \not\preceq b$ and $b \not\preceq a$, respectively). This Power Question largely defines and explores a partial ordering on the set of binary partitions of n.

The notation $a \prec b$ means that $a \preceq b$ and $a \neq b$. The symbols $\succ$ and $\succeq$ may also be used, and they are defined in the following way: $a \succ b$ means $b \prec a$, and $a \succeq b$ means $b \preceq a$.

If a and b are elements of S with $a \prec b$, and if there is no element $c \in S$ for which $a \prec c \prec b$, then it is said that b *covers* a.

Let a denote the binary partition $(a_1, a_2, \ldots, a_n)$. Similarly, let b denote the binary partition $(b_1, b_2, \ldots, b_n)$. In this Power Question, define $a \prec b$ if it is possible to obtain a from b by a sequence of replacing one 2^k by two 2^{k-1}s (and deleting a 0). For example, $(4,1,1,1,1,0,0,0) \prec (4,4,0,0,0,0,0,0)$ because $(4,1,1,1,1,0,0,0) \prec (4,2,1,1,0,0,0,0) \prec (4,2,2,0,0,0,0,0) \prec (4,4,0,0,0,0,0,0)$. This Power Question will use this partial ordering on binary partitions.

4. **a.** Show that the binary partitions of 5 are totally ordered; i.e., if p and p' are two different binary partitions of 5, then either $p \prec p'$ or $p' \prec p$. [2 pts]

b. Show that the binary partitions of 8 are not totally ordered, i.e., find two binary partitions of 8 – call them q and q' – such that $q \not\prec q'$ and $q' \not\prec q$. [3 pts]

5. **a.** Find the smallest binary partition of n, using this partial ordering. That is, find the binary partition p such that for all other binary partitions p', $p \prec p'$. [2 pts]

b. Find the largest binary partition of n, using this partial ordering. That is, find the binary partition P such that for all other binary partitions P', $P \succ P'$. [3 pts]

Hasse Diagrams

Suppose that a set S has a partial ordering $\preceq$. Then a *Hasse diagram* can be used to display the covering relation in a graphical way. The Hasse diagram is a graph whose vertices are the elements of S and where edges are drawn between two elements x and y if $x \prec y$ or $y \prec x$ and if there is no element z for which $x \prec z$ and $z \prec y$ or for which $y \prec z$ and $z \prec x$. Also, y appears "above" x if $x \prec y$. Note that it is possible for more than one element to appear on the same level of a Hasse diagram. For example, the partially ordered set of divisors of 12, ordered by divisibility, is shown in Figure 1. In this diagram, the number 1 is said to be at Level 0 because it is the least divisor in the partial ordering shown. The numbers 2 and 3 are said to be on Level 1 because $2 \succ 1$ and $3 \succ 1$ and there is no number n such that $2 \succ n$ and $n \succ 1$ or $3 \succ n$ and $n \succ 1$. Other Levels are similarly defined. The number 12 is on Level 3.

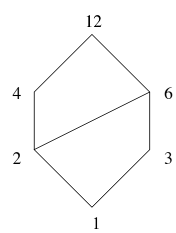


Figure 1

Hasse diagrams can be drawn to show the ordering of partitions such as the ones from Problems 4 and 5. For convenience, rather than labeling the vertices in the Hasse diagram with the partition itself, like $(8, 0, 0, 0, 0, 0, 0, 0)$, it is common to label the vertex with its nonzero parts only, using exponents to indicate parts within the partition with multiplicity greater than 1. For example, the partition $(8, 0, 0, 0, 0, 0, 0, 0)$ would be labeled 8, the partition $(4, 2, 2, 0, 0, 0, 0, 0)$ would be labeled 42^2, and the partition $(2, 2, 1, 1, 1, 1, 0, 0)$ would be labeled $2^2 1^4$.

6. **a.** Draw the Hasse diagram for the binary partitions of 8. Label each vertex. [1 pt]

b. List a path through the Hasse diagram for the binary partitions of 8 that begins at the bottom vertex (that is, the vertex at Level 0) and, traveling only along edges, passes through every other vertex exactly once. Such a path is called a *Hamiltonian path*. [1 pt]

c. Let n be an even integer with $n > 4$. Let S be the set of binary partitions of n. Let S_1 be the set of binary partitions of $\frac{n}{2}$. Let S_2 be the set of binary partitions of $n - 2$. Prove that there is a bijection B (i.e., a one-to-one correspondence) from the set S to the set $S_1 \cup S_2$. Prove that this bijection B preserves order; that is, given that $p \prec p'$ for binary partitions $p, p' \in S$, then either $B(p) \prec B(p')$ or $B(p)$ and $B(p')$ are incomparable. [3 pts]

d. Prove that for each positive integer n, the Hasse diagram of the binary partitions of n has a Hamiltonian path that begins with the vertex at Level 0. [5 pts]

Let $f_L(n)$ represent the number of elements at level L in the Hasse diagram of the binary partitions of n.

7. Prove that the value of $f_L(n)$ is the number of binary partitions of n that have $n - L$ nonzero parts. [3 pts]

8. Not all partitions are binary partitions. Some partitions have parts of the form $2^j - 1$ where j is a nonnegative integer. Such partitions will be called *s-partitions*, and their parts are written in nonincreasing order. Two s-partitions of 5 are $(3, 1, 1, 0, 0)$ and $(1, 1, 1, 1, 1)$.

- **a.** List two partitions of 7, one that is an s-partition and one that is neither an s-partition nor a binary partition. Make sure to identify which is which. [2 pts]
- **b.** Prove that if $n \geq 2L$, the value of $f_L(n)$ is equal to the number of s-partitions of L. [3 pts]

Trinary Partitions

A *trinary partition* of a positive integer n is an ordered n-tuple of non-increasing integers, each of which is either 0 or a power of 3, whose sum is n. Let $p_3(n)$ denote the number of trinary partitions of n. For example, $p_3(4) = 2$ because of the two ordered quadruples $(3, 1, 0, 0)$ and $(1, 1, 1, 1)$.

As with binary partitions, one can define partial orderings for trinary partitions. Let c denote the trinary partition $(c_1, c_2, \ldots, c_n)$. Similarly, let d denote the trinary partition $(d_1, d_2, \ldots, d_n)$. Define $c \prec d$ if it is possible to obtain c from d by a sequence of replacing one 3^k by three 3^{k-1}s (and deleting two 0s).

9. Draw the Hasse diagram for the trinary partitions of 12. Label each vertex. [2 pts]

10. State a value of n less than 23 for which the Hasse diagram of the trinary partitions of n does **not** contain a Hamiltonian path. Prove your claim. (Recall that a Hamiltonian path is defined in Problem 6b.) [6 pts]

5 Solutions to Power Question

1. The values are as follows.
The value of $p_2(4)$ is **4** from $(4,0,0,0)$, $(2,2,0,0)$, $(2,1,1,0)$, and $(1,1,1,1)$.
The value of $p_2(5)$ is **4** from $(4,1,0,0,0)$, $(2,2,1,0,0)$, $(2,1,1,1,0)$, and $(1,1,1,1,1)$.
The value of $p_2(6)$ is **6** from $(4,2,0,0,0,0)$, $(4,1,1,0,0,0)$, $(2,2,2,0,0,0)$, $(2,2,1,1,0,0)$, $(2,1,1,1,1,0)$, and $(1,1,1,1,1,1)$.
The value of $p_2(7)$ is **6** from $(4,2,1,0,0,0,0)$, $(4,1,1,1,0,0,0)$, $(2,2,2,1,0,0,0)$, $(2,2,1,1,1,0,0)$, $(2,1,1,1,1,1,0)$, and $(1,1,1,1,1,1,1)$.

2. **a.** A binary partition for n can be changed into a binary partition for $n+1$ by inserting a 1 immediately following the rightmost nonzero entry. For example, $(2,2,1,1,0,0)$ is a binary partition of 6, and this can be changed into $(2,2,1,1,1,0,0)$, which is a binary partition of 7. Similarly, $(1,1,1,1)$ becomes $(1,1,1,1,1)$, and these are binary partitions of 4 and 5, respectively. Different binary partitions of n become different binary partitions of $n+1$ in this way. Because every binary partition of n maps to a binary partition of $n+1$, there are at least as many binary partitions of $n+1$ as of n.

b. The answer to the question is **no**. If n is even, then $n+1$ is odd. Because a binary partition of an odd number must contain at least one 1, the correspondence described in the solution to Problem 2a is one-to-one. To establish this, given a binary partition of $n+1$, delete one 1 to obtain a binary partition of n. Thus $p_2(n+1) = p_2(n)$ for any even n. However, note that if n is odd, then $p_2(n+1)$ is strictly greater than $p_2(n)$ because in addition to the binary partitions of $n+1$ that can be obtained by inserting a 1 into a binary partition of n, there are binary partitions of $n+1$ consisting of all even numbers.

3. **a.** Let n be an even integer. Given a binary partition of n, either all of its parts are even or there is at least one 1. If all parts are even, then all of the positive parts must occur in the first $\frac{n}{2}$ elements of the n-tuple (otherwise, their sum would exceed n). Dividing all the parts by 2 yields an n-tuple of powers of 2 and 0s that sum to $\frac{n}{2}$. Note that the last $\frac{n}{2}$ elements of this n-tuple must all be 0s and truncating them results in an $\left(\frac{n}{2}\right)$-tuple which is a binary partition of $\frac{n}{2}$. Conversely, for any binary partition of $\frac{n}{2}$, by doubling each element and appending $\frac{n}{2}$ zeros at the end results in an n-tuple which is a binary partition of n. Hence $p_2\left(\frac{n}{2}\right)$ counts the number of binary partitions of n in which all the parts are even. If there is at least one 1, then the assumption that n is even means that there are at least two 1s, and deleting them produces a binary partition of $n-2$. Conversely, for each binary partition of $n-2$, appending two 1s (right before the 0s, if there are any, otherwise, append them to the end of the $(n-2)$-tuple) produces an n-tuple that is a binary partition of n. Hence $p_2(n-2)$ counts the number of binary partitions of n with at least one 1, and thus the desired recursion follows.

b. **There is no such n.** Proceed by cases. If n is odd, then $p_2(n) = p_2(n-1)$ because all binary partitions of an odd number include at least one 1, and removing that 1 gives a bijection to partitions of $n-1$. If n is even, then $p_2(n) = p_2\left(\frac{n}{2}\right) + p_2(n-2)$ by Problem 3a. Because $p_2(2) = 2$ and $p_2(4) = 4$, it follows inductively that $p_2(n)$ is even for all $n > 1$.

4. **a.** The four binary partitions of 5 are ordered in the following way: $(1,1,1,1,1) \prec (2,1,1,1,0) \prec (2,2,1,0,0) \prec (4,1,0,0,0)$. This is a total ordering.

b. Consider these two binary partitions of 8: $(4,1,1,1,1,0,0,0)$ and $(2,2,2,2,0,0,0,0)$. The 4 can break down into two 2s, but that is the only way to generate 2s, and that does not generate four 2s, so it is not the case that $(2,2,2,2,0,0,0,0) \prec (4,1,1,1,1,0,0,0)$. Likewise, it is not possible for a partition with no 4s to be related by $\prec$ to a partition with 4s, so it is not the case that $(4,1,1,1,1,0,0,0) \prec (2,2,2,2,0,0,0,0)$.

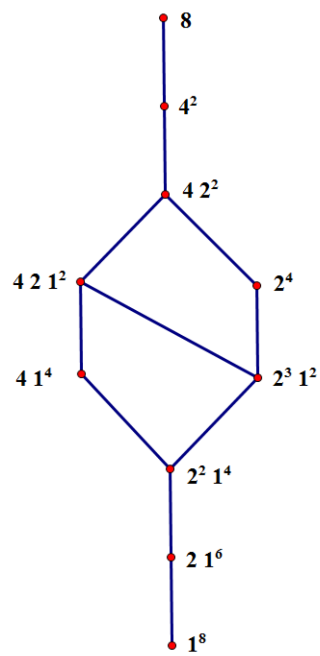
Similarly, the following two pairs of partitions are not comparable:

$$(4,1,1,1,1,0,0,0) \text{ and } (2,2,2,1,1,0,0,0), \qquad (4,2,1,1,0,0,0,0) \text{ and } (2,2,2,2,0,0,0,0).$$

5. **a.** The smallest binary partition of n is the partition with n 1s. That is, it is the partition $(1,1,\ldots,1)$ with n copies of 1 in the partition. If a partition has any part that is not equal to 0 or 1, then that part is a positive power of 2 that may be replaced by lesser powers of 2. This process can continue until all of the powers of 2 are 1s, but the process cannot continue beyond that because a partition must consist only of integers. Thus every binary partition is greater than the one consisting of all 1s. Also note that the partition $(1, 1, \ldots, 1)$ is not larger than or incomparable with any other partition. Thus $(1, 1, \ldots, 1)$ is the smallest binary partition.

b. The largest binary partition of n is the one that essentially gives the base-2 equivalent of n. That is, if the base-2 representation of n has 1s in the 2^a, 2^b, 2^c places, and so on, with $a > b > c > \cdots$, then the largest binary partition of n is $(2^a, 2^b, 2^c, \ldots, 0, 0, 0)$ (with enough 0s to finish the partition). If a partition contains two copies of a number other than 0, they may be combined to form a greater power of 2. Thus, starting with any partition, continue combining powers of 2 until the partition contains no repeated numbers. Because no further combinations are allowed at this point, by the definition, there can be no partition larger than a partition with no repeated numbers. *A priori* there could be other partitions that are not comparable to such a partition, however. But because such a partition expresses n as a sum of powers of 2, with no repeated nonzero parts, it is essentially the base-2 representation of n, which is unique. So because any partition may undergo the process of combining like powers of 2 until there are no repeats, every binary partition is less than the unique partition which expresses n in binary.

6. **a.** A Hasse diagram for the binary partitions of 8 is shown below.



b. The following sequence of vertices is a Hamiltonian path: $1^8, 21^6, 2^21^4, 41^4, 421^2, 2^31^2, 2^4, 42^2, 4^2, 8$. This path begins at the vertex at Level 0, then visits each vertex exactly once.

c. The bijection was effectively established in the solution to 3a. Now consider whether $B(p) \prec B(p')$ for p and p' which are different binary partitions of n and for which $p \prec p'$. Suppose first that both p and p' have no 1s at all. Then dividing all parts by 2 preserves the covering relation. This is because p can be produced by swapping out powers of 2 in p' for lesser powers of 2, and dividing all parts by 2 does not change this covering. Now suppose that p and p' both have at least two 1s. Then removing two 1s from each partition preserves the covering relation. This is because removing two 1s from the partition in no way changes the "swap-outs" of powers of 2 for lesser powers of 2. Now suppose that one of p or p' has two 1s and the other doesn't. Then $B(p)$ is incomparable with $B(p')$, because $B(p) \in S_1$ and $B(p') \in S_2$ or vice versa, and these are sets of partitions of different numbers because $n - 2 > \frac{n}{2}$ when $n > 4$. The relation $\prec$ is not defined between partitions of different numbers, so $B(p)$ and $B(p')$ are not comparable.

d. As in the solutions to Problems 2 and 3, it is only necessary to consider the problem for even n. This is because a Hasse diagram for partitions of $2r + 1$ is the same as that for the partitions of $2r$ with an extra 1 in every partition. So let $n = 2r = a \cdot 2^k$, where a is an odd integer and k is positive. The proof proceeds by strong induction to prove that there is a Hamiltonian path through the Hasse diagram for partitions of n starting at 1^n and ending at $(2^k)^a$. To get started, the path $1^2 \to 2$ is clearly a Hamiltonian path through the vertices of the Hasse diagram for partitions of 2, and the path $1^4 \to 21^2 \to 2^2 \to 4$ is a Hamiltonian path through the vertices of the Hasse diagram for partitions of 4. These establish the base case.

The idea of the proof will be to use the bijection from part c. In essence, the partitions of n will be split into those that are matched with partitions of $n-2$ and those that are partitions of $n/2$, and the Hamiltonian paths through each of these sets of smaller partitions will be joined into a single path through the partitions of n.

The path starts at 1^n. Consider this as $1^{n-2}1^2$. That is, group the first $n-2$ 1s together and the last two 1s together. From the induction hypothesis, there is a Hamiltonian path through the partitions of $n-2$ starting with 1^{n-2}. By adjoining two 1s to each of these, there is a Hamiltonian path through the partitions of n that have two or more 1s in them. This path ends at $(2^\ell)^c 1^2$, where $n-2 = 2(r-1) = c \cdot 2^\ell$, c is an odd integer, and ℓ is positive. The manner in which the path will be continued through the partitions of n with no 1s in them is dependent on the parity of r.

First consider the case in which r is even. Then $r-1$ is odd. In this case, $c = r-1$ and $\ell = 1$. Thus the path in the previous paragraph ended at $2^c 1^2$. The next step in the path will be to $2^{c+1} = 2^r$. Now by induction, there is a Hamiltonian path through the partitions of $n/2 = r$ starting at 1^r and ending at $(2^{k-1})^a$. Double every entry of each of these partitions. This gives a Hamiltonian path through the partitions of n that have no 1s in them, starting at 2^r and ending at $(2^k)^a$. Attaching this to the end of the path from 1^n to 2^r already obtained results in the desired Hamiltonian path from 1^n to $(2^k)^a$.

In the case where r is odd, $r-1$ is even. In this case, $a = r$ and $k = 1$, while $n-2 = 2(r-1)$ must be a multiple of 4. So the path through the partitions of n containing 1s ended at $(2^\ell)^c 1^2$ for some $\ell \geq 2$. The next step will be to the partition $(2^\ell)^c 2$. Now $r = n/2$ is odd, so every binary partition of r must contain at least one 1. Note that these are the binary partitions of $r-1$ with a 1 adjoined, and by induction, there is a Hamiltonian path through them starting at 1^r and ending at $(2^{\ell-1})^c 1$. Double each entry of each of these, and run through this path *backward* to obtain a path starting at $(2^\ell)^c 2$ and ending at 2^r. Concatenating this with the path from 1^n to $(2^\ell)^c 2$ previously obtained will yield the desired path from 1^n to $2^r = (2^k)^a$.

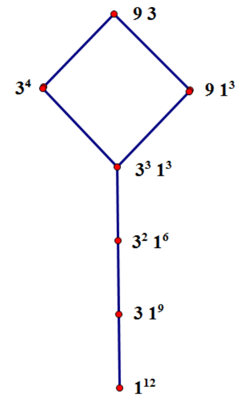
7. Proceed by induction on L. The only element for which $L = 0$ is the smallest binary partition of n, which indeed has n 1s by the solution to Problem 5, and so this partition has $n - L = n - 0$ parts. Moving up via a covering relation, when two 2^ks are replaced by one 2^{k+1}, it is true that one nonzero part is lost. Assuming an element at level L has $n - L$ nonzero parts, an element at level $L+1$ will have $n - L - 1 = n - (L+1)$ parts. This establishes the inductive step.

8. **a.** There are several s-partitions of 7. They include $(7,0,0,0,0,0,0)$, $(3,3,1,0,0,0,0)$, $(3,1,1,1,1,0,0)$, and $(1,1,1,1,1,1,1)$. Other partitions of 7 are not s-partitions; some of those are also not binary. They include $(6,1,0,0,0,0,0)$, $(5,2,0,0,0,0,0)$, $(5,1,1,0,0,0,0)$, $(3,2,2,0,0,0,0)$, and $(3,2,1,1,0,0,0)$.

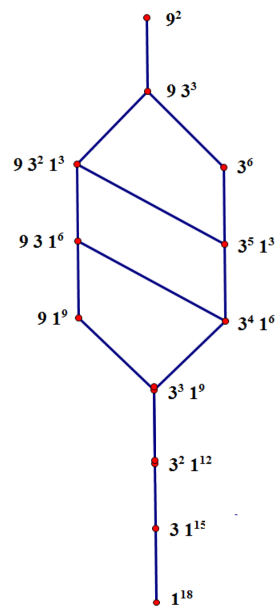
b. Construct a bijection. Given a binary partition of n at level L, subtract 1 from each nonzero part. Hence each part 2^j is replaced by $2^j - 1$, leaving an s-partition of $n - (n - L) = L$.

For the inverse map, consider an s-partition of L, with parts of the form $2^j - 1$ for some positive integer j. This cannot have more than L parts, and that is achievable only with all parts equal to 1. Consider the partition to end in enough 0s to bring the total number of parts up to $n - L$; this is possible because $n - L = n - 2L + L \geq 2L - 2L + L = L$. Now add 1 to each part. The result is a binary partition of $L + n - L = n$ with $n - L$ parts, which puts it at level L by Problem 7.

9. A Hasse diagram for the trinary partitions of 12 is shown below.



10. The three possible answers are **18** or **19** or **20**. A Hasse diagram for the trinary partitions of 18 is shown below. The answer will be established when it is shown that there is no Hamiltonian path through the Hasse diagram. Note that all Hasse diagrams for the trinary partitions of 18 are isomorphic to the one in the figure. Note also that the Hasse diagram for the trinary partitions of 19 is isomorphic to the Hasse diagram for the trinary partitions of 18; a bijection can be established by adding a 1 after the rightmost nonzero part of any trinary partition of 18. Similarly, the Hasse diagram for the trinary partitions of 20 is isomorphic to the Hasse diagram for the trinary partitions of 18.



Suppose there is a Hamiltonian path through the Hasse diagram. Because the Hasse diagram has only one edge coming from 1^{18} and only one edge coming from 9^2, the Hamiltonian path must begin and end at those vertices in some order. Without loss of generality, assume the path begins at 1^{18} and ends at 9^2.

The path must be of the form $1^{18}, 31^{15}, 3^21^{12}, 3^31^9, \ldots, 9^2$. The first choice comes when deciding if the fifth vertex in the path is 91^9 or 3^41^6. If the choice is to make 3^41^6 the fifth vertex in the path, then it is not possible to enter and leave 91^9 because the edge $\{3^31^9, 91^9\}$ cannot be used for entry or for exit; both would require revisiting 3^31^9. So the fifth vertex in the path must be 91^9, and thus the sixth vertex in the path is 931^6.

The next choice comes when considering the seventh vertex in the path. If the seventh vertex is 93^21^3, then it will be impossible to visit 3^41^6 and also 9^2. If the seventh vertex is 3^41^6, then the eighth vertex in the path is 3^51^3, and either choice from that vertex (93^21^3 or 3^6) leads to a contradiction because to visit the other requires a path that prevents visiting 9^2. This completes the proof that the Hasse diagram of the trinary partitions of 18 does not contain a Hamiltonian path.

Checking other possible values of n that are multiples of 3, there is a Hamiltonian path through the Hasse diagrams for each one, as the following argument demonstrates.

- A Hamiltonian path through the Hasse diagram for $n = 3$ is $1^3, 3$.
- A Hamiltonian path through the Hasse diagram for $n = 6$ is $1^6, 31^3, 3^2$.
- A Hamiltonian path through the Hasse diagram for $n = 9$ is $1^9, 31^6, 3^21^3, 3^3, 9$.
- A Hamiltonian path through the Hasse diagram for $n = 12$ is $1^{12}, 31^9, 3^21^6, 3^31^3, 91^3, 93, 3^4$.
- A Hamiltonian path through the Hasse diagram for $n = 15$ is $1^{15}, 31^{12}, 3^21^9, 3^31^6, 91^6, 931^3, 93^2, 3^5, 3^41^3$.
- A Hamiltonian path through the Hasse diagram for $n = 21$ is $1^{21}, 31^{18}, 3^21^{15}, 3^31^{12}, 91^{12}, 931^9, 3^41^9, 3^51^6, 93^21^6, 93^31^3, 93^4, 9^23, 9^21^3$.

Note that there are Hamiltonian paths through the Hasse diagram for both $n = 1$ (the path is just the vertex 1) and $n = 2$ (the path is $1^2, 2$). Note also that if k is a positive integer, the Hasse diagrams for the trinary partitions of $n = 3k$, $n = 3k + 1$, and $n = 3k + 2$ are all isomorphic, in the same way that the Hasse diagrams for the trinary partitions of 18, 19, and 20 are isomorphic.

Thus the only values of n less than 23 for which the Hasse diagram of trinary partitions does not contain a Hamiltonian path are 18 or 19 or 20.