April 6, 2017
Round I: Arithmetic & Number Theory

1. (1 point) Find the difference, in seconds, between two and a half minutes and 10% of an hour.

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10% of an hour is 6 minutes. The difference between 6 min. and 2 ½ min. is 3 ½ min. 3 ½ min is equal to 210 seconds.

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2. (2 points) Suppose that A, B, C are positive integers such that $A+\dfrac{1}{B+\dfrac{1}{C}}=\dfrac{29}{13}$.

Compute $A+B+C$.

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Since A, B, C are positive integers, $0<\dfrac{1}{B+\dfrac{1}{C}}<1$. Also, $A+\dfrac{1}{B+\dfrac{1}{C}}=2\dfrac{3}{13}$. So, $A=2$ and

$\dfrac{1}{B+\dfrac{1}{C}}=\dfrac{3}{13}$, meaning that $B+\dfrac{1}{C}=\dfrac{13}{3}=4\dfrac{1}{3}$. So, $B=4$ and $C=3$. Finally,
$A+B+C=2+4+3=9$.

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3. (3 points) Find the sum of all positive two digit numbers that have the property that the value of the number is the same as three times the product of the digits in the number.

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$$10t+u=3tu\rightarrow 10t=3tu-u=u(3t-1)\rightarrow u=\frac{10t}{3t-1}.$$

Now try different values of $t$ and check the results.

$t=1\rightarrow u=5\rightarrow 15,\ t=2\rightarrow u=4\rightarrow 24$

$t=3\rightarrow u=\dfrac{30}{8}, t=4\rightarrow u=\dfrac{40}{11}, t=5\rightarrow u=\dfrac{50}{14}$

$t=6\rightarrow u=\dfrac{60}{17}, t=7\rightarrow u=\dfrac{70}{20}, t=8\rightarrow u=\dfrac{80}{25}$

$t=9\rightarrow u=\dfrac{90}{28}$

The only numbers that we will consider are 15 and 24 = **39**. 15 = 3(1)(5) and 24=3(2)(4).

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April 6, 2017

Round II: Algebra I (Real numbers and no transcendental functions)

1.(1 point) Solve: $\left(\sqrt[3]{x-2}\right)^2 = 4$.

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$$\left(\sqrt[3]{x-2}\right)^2 = 4 \rightarrow \sqrt[3]{x-2} = \pm 2 \rightarrow x-2 = \pm 8 \rightarrow x = 10 \text{ or } -6.$$

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2. (2 points) Solve for x: $\dfrac{x^{-2}+1}{x^{-3}-x^{-4}} = \dfrac{-5x}{x^{-1}-1}$.

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$$LHS = \left(\frac{\frac{1}{x^2}+\frac{1}{1}}{\frac{1}{x^3}-\frac{1}{x^4}}\right) * \frac{x^4}{x^4} \rightarrow \frac{x^2+x^4}{x-1}; RHS = \left(\frac{-5x}{\frac{1}{x}-1}\right) * \frac{x}{x} \rightarrow \frac{-5x^2}{1-x}$$

$$\frac{x^2+x^4}{x-1} = \frac{-5x^2}{1-x} \rightarrow x^2+x^4 = 5x^2$$

$$x \neq 0,1 \rightarrow 4x^2 - x^4 = 0 \rightarrow x^2\left(4-x^2\right) = 0$$

$$x \neq 0, x = \pm 2 \rightarrow x = +2, x = -2$$

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3. (3 points) Let S be the set of real numbers y such that $\sqrt{y} = 5\sqrt{x} - 2x$ for some real number x. Write S using interval notation.

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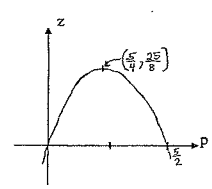
Let $p = \sqrt{x}, z = 5p - 2p^2$. Note that $p \ge 0$.

$$z = p(5-2p). \text{ So, } 0 \le \sqrt{y} \le \frac{25}{8} \rightarrow 0 \le y \le \frac{625}{64} \rightarrow \left[0, \frac{625}{64}\right]$$

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Round III: Geometry (figures are not necessarily drawn to scale)

1. (1 point) Determine the number of sides of a regular polygon where the sum of two of its interior angles is $336^\circ$.

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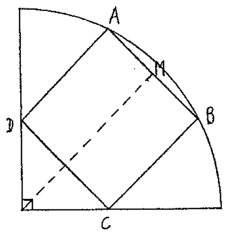
If 2 of the angles add to 336, then each angle (the polygon is regular) is $168^\circ$. The corresponding exterior angle is $12^\circ$ and the sum of all the exterior angle of a regular polygon is $360^\circ$, so $360 \div 12 = 30$ sides.

2. (2 points) The diagram shows square *ABCD* inscribed in a 90° sector of a circle. Points *A* and *B* lie on the circle and points *C* and *D* lie on the perpendicular radii. *M* is the midpoint of *AB*. Each side of the square has length 6 meters, and the area of the region bounded by the entire circle is $k\pi$ square meters. Find the value of *k*.

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Draw a radius from O to B. Draw the 2 diagonals of the square. Since the sides of the square are each 6 meters, the diagonals are $6\sqrt{2}$. This gives the coordinates of B as $\left(6\sqrt{2}, 3\sqrt{2}\right)$. This leads to

$$r = \sqrt{\left(6\sqrt{2}\right)^2 + \left(3\sqrt{2}\right)^2} = \sqrt{72+18} = \sqrt{90}$$

$$A = \pi r^2 = \pi\left(\sqrt{90}\right)^2 = 90\pi \rightarrow\rightarrow K = 90$$

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3. (3 points) Let $\overline{AB}$ be a diameter of a circle and C be a point on $\overline{AB}$ with $2 \bullet AC = BC$. Let D and E be points on the circle such that $\overline{DC} \perp \overline{AB}$ and $\overline{DE}$ is a second diameter. What is the ratio of the area of triangle DCE to the area of triangle ABD?

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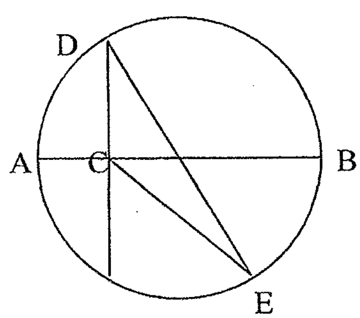


Let O be the center of the circle. Both ΔDCE and ΔABD Have a diameter as one of its sides, therefore the ratio of their areas is the ratio of 2 altitudes to the diameter.

These altitudes are $\overline{DC}$ and the altitude from C to $\overline{DO}$ in ΔDCE. Let F be the foot of the second altitude. Since $\Delta CFD \sim \Delta DCE$

$$\rightarrow \frac{CF}{DC} = \frac{CO}{DO} = \frac{AD - AC}{DO} = \frac{\frac{1}{2}AB - \frac{1}{3}AB}{1} = \frac{1}{3}$$

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April 6, 2017
Round IV: Algebra II

1. (1 point) Find the value of $\log_2\left(\sqrt{21}-\sqrt{5}\right)+\log_2\left(\sqrt{21}+\sqrt{5}\right)$.

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$$\log_2\left(\sqrt{21}-\sqrt{5}\right)+\log_2\left(\sqrt{21}+\sqrt{5}\right) = \log_2\left(\sqrt{21}-\sqrt{5}\right)\left(\sqrt{21}+\sqrt{5}\right) = \log_2\left(21-5\right) = \log_2\left(16\right) = 4$$

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2. (2 points) Find the largest x so that $|x| < \frac{1}{2}$ and $x^4 + x^3 - 3x^2 + x = 0$.

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$x^4 + x^3 - 3x^2 + x = 0 \rightarrow x\left(x^3 + x^2 - 3x + 1\right) = 0$ This last factor has a root of +1. Using synthetic division we will attain $x(x-1)(x^2 + 2x - 1) = 0 \rightarrow x = 0, 1, -1 \pm \sqrt{2}$. Now compare with the other restriction: $x = 0, x \neq 1, x \neq -1 - \sqrt{2}, x = -1 + \sqrt{2}$. The largest value left is $x = -1 + \sqrt{2}$.

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3. (3 points) The parabola $y = ax^2 + bx + c$ intersects the y-axis at (0, 8) and the x-axis in a single point (2, 0). How many pairs of integers (m, n) with -2000 < n < 2000 lie on the parabola?

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We know the parabola is in the form $y = a\left(x-2\right)^2$ with $8 = a\left(0-2\right)^2 \rightarrow a = 2 \rightarrow y = 2\left(x-2\right)^2$.

$2\left(x-2\right)^2 \leq 2000 \rightarrow \left(x-2\right)^2 \leq 1000$

$-\sqrt{1000} \leq x - 2 \leq \sqrt{1000} \rightarrow -31 \leq x - 2 \leq 31 \rightarrow -29 \leq x \leq 33$

Since the problem asks for integers, we use $\sqrt{1000} \approx 31$. Now, giving us a total 29+1+33 = 63 points.

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## Round V: Analytic Geometry

1. (1 point) Let $f(x) = x^2 + 10x + 5$. Express, as an ordered pair, the vertex of the graph of $y = f^{-1}(x)$.

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$f(x) = x^2 + 10x + 25 - 20 \rightarrow\rightarrow f(x) = \left(x+5\right)^2 - 20$. The vertex of this parabola is (-5, -20). The vertex of its inverse is (-20, -5).

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2. (2 points) Q is the point of the circle $x^2 - 10x + y^2 + 6y + 29 = 0$ which is furthest from the point P(-1, -6). Determine the distance PQ.

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Compete the square to put the equation into an easier form to work with.

$$\left(x^2 - 10x + 25\right) + \left(y^2 + 6y + 9\right) = -29 + 25 + 9 \rightarrow \left(x-5\right)^2 + \left(y+3\right)^2 = 5$$

This is a circle with center: (5, -3) with a radius of $\sqrt{5}$. The distance from P to Q includes the distance from the center plus the radius from the other side of the circle.

$$PQ = \sqrt{(5+1)^2 + (-3+6)^2} + \sqrt{5}$$
$$= \sqrt{36+9} + \sqrt{5} = \sqrt{45} + \sqrt{5} = 3\sqrt{5} + \sqrt{5} = 4\sqrt{5}$$

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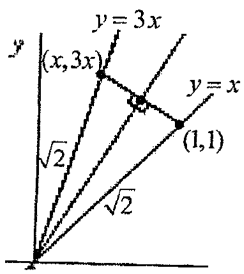
3. (3 points) The lines given by $y = x$ and $y = 3x$ form an acute angle in the first quadrant. What is the slope of the line that bisects that angle?




Draw a line segment that is perpendicular to the bisector to create two congruent triangles, as shown. Taking the point (1, 1) on the line y = x yields a length of $\sqrt{2}$ for the hypotenuse of the triangle. Thus, the other hypotenuse must also have length $\sqrt{2}$. Now determine the point on the line y = 3x, labeled (x, 3x): $x^2 + (3x)^2 = \left(\sqrt{2}\right)^2 \to 10x^2 = 2 \to x^2 = \frac{1}{5}$

$\to x = \frac{\sqrt{5}}{5}$., taking the positive value of x, which makes y= $\frac{3\sqrt{5}}{5}$.

The midpoint of the segment, which is on the bisector can now be found: $\left(\frac{1+\sqrt{5}/5}{2}, \frac{1+3\sqrt{5}/5}{2}\right)$.

So the slope of the bisector is: $\dfrac{\frac{1+3\sqrt{5}/5}{2} - 0}{\frac{1+\sqrt{5}/5}{2} - 0} = \frac{5+3\sqrt{5}}{5+\sqrt{5}} = \frac{1+\sqrt{5}}{2}$.

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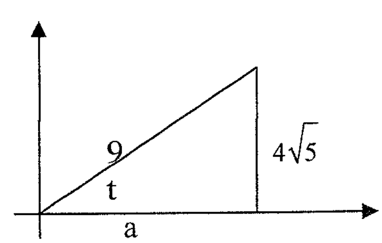
April 6, 2017
Round VI: Trigonometry

1. (1 point) If $\sin t = \frac{4\sqrt{5}}{9}; 0^o < t < 90^o$, compute the value of $\log_3 \sin t + \log_3 \cot t$.

$\log_3 \sin t + \log_3 \cot t = \log_3 (\sin t \cdot \cot t) = \log_3 \cos t$

$a^2 + \left(4\sqrt{5}\right)^2 = 9^2 \to a^2 + 80 = 81$

$a = +1, a \neq -1 \to \cos t = \frac{1}{9} \to\to \log_3 \left(\frac{1}{9}\right) = -2$

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2. (2 points) Suppose that $\tan 2x = \frac{5}{12}$. Find all the possible values of $\tan^2 x + \cot^2 x$.

Using $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$, we have

$$\frac{5}{12} = \frac{2\tan x}{1-\tan^2 x} \to 5 - 5\tan^2 x = 24\tan x$$

$$\to 5\tan^2 x + 24\tan x - 5 = 0 \to \left(5\tan x - 1\right)\left(\tan x + 5\right) = 0$$

$$\tan x = \frac{1}{5}, -5 \to \cot x = 5, -\frac{1}{5} \to \tan^2 x + \cot^2 x = \frac{1}{25} + 625 = \frac{626}{25}$$

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3. (3 points)

The equation $\left(2\sin x - \sqrt{3}\right)^2 + \left(2\cos x - 1\right)^2 + \left(\tan x - \sqrt{3}\right)^2 = 0$ has exactly 8 solutions in the range of $0^o < x \le \mu^o$. Find the smallest possible value of μ.

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Each term is non negative and therefore each term must be zero to product a sum of zero. Thus, the first terms requires x to be coterminal with 60° or 120°, the 2nd term requires x to be coterminal with 60° or 300°, and the 3rd term requires x to be coterminal with 60° and 240°. Therefore, looking for common ground $\to 60 + 360n$.

The first solution must be $60 + 360(0) = 60$ and the 8th solution must be $60 + 360(7) = 2580$.

So $\mu = 2580$

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TEAM ROUND NO CALCULATORS

1) Let N be a 3-digit base ten positive integer whose middle digit is 0. N is a multiple of 11 and the quotient $\dfrac{N}{11}$ equals the sum of the squares of the digits of N. Determine N.

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Let the 3-digit integer be of the form XOY with $x - 0 + y = 11 \to y = 11 - x$. Now set up the rest of the given information:

$$\frac{100x + y}{11} = x^2 + y^2 \to 100x + (11-x) = 11(x^2 + (11-x)^2)$$

$$\to 99x + 11 = 11x^2 + 11(11-x)^2 \to 9x + 1 = x^2 + (11-x)^2$$

$$\to 2x^2 - 31x + 120 = 0 \to \left(2x - 15\right)\left(x - 8\right) = 0$$

$$x = \cancel{\frac{15}{2}}, 8 \to y = 3 \to N = 803$$

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2) Determine a value for c.

$$\begin{cases} \dfrac{ef-2d}{5}=18 \\ \dfrac{8cf+10d}{10f}=22 \\ \dfrac{9d-6cf}{15f}=54 \end{cases} \qquad \begin{cases} 4cf+5d=22(5f) \\ 3d-2cf=54(5f) \end{cases} \rightarrow \begin{cases} 4cf+5d=110f \\ -4cf+6d=540f \end{cases}$$

$$\rightarrow 11d = 650f \rightarrow d = \frac{650}{11}f$$

Now, substitute into 3[rd] equation $3\left(\frac{650}{11}\right)f - 2cf = 270f \rightarrow c = -\frac{504}{11}$.

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3) In $\Delta ABC$, $AB = AC = 65$ and the area of triangle ABC is 1848. Find the maximum possible length of the altitude from A to $\overline{BC}$.

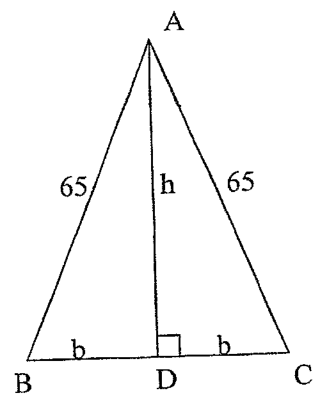
$$\frac{1}{2}(2b)h = 1848 \rightarrow bh = 1848 \text{ and } b^2 + h^2 = 65^2 = 4225$$

$$(b+h)^2 = b^2 + 2bh + h^2 = 7921 = 89^2 \rightarrow b + h = 89$$

$$b = 89 - h \rightarrow h(89-h) = 1848$$

$$h^2 - 89h + 1848 = 0 \rightarrow (h-33)(h-56) = 0$$

$$h = 56$$

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4) Determine the sum of the y-coordinates of the three points of the intersection of $y = x^2 - x - 5$ and $y = \frac{1}{x}$.

At the points of intersection, $\frac{1}{x} = x^2 - x - 5$, so $1 = x^3 - x^2 - 5x$, so $x^3 - x^2 - 5x - 1 = 0$. Letting the roots of this equation be $r, s,$ and $t$, we require $\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = \frac{rs + st + rt}{rst}$.

It is known that if the roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$ are $r, s,$ and $t$, then $r + s + t = -\frac{b}{a}$, $rs + st + rt = \frac{c}{a}$, and $rst = -\frac{d}{a}$. So, here, $rs + st + rt = -5$ and $rst = 1$, giving $\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = -5$.

5) If the line $y = mx + 1$ intersects the conic $x^2 + 4y^2 = 1$ exactly once, find all possible values of $m$.

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Substitute for y in the ellipse.

$$x^2 + 4(mx+1)^2 = 1 \rightarrow x^2 + 4(m^2x^2 + 2mx + 1) = 1 \rightarrow (4m^2 + 1)x^2 + 8mx + 3 = 0$$

To be true,

$$\text{"}b^2 - 4ac = 0\text{"} \rightarrow (8m)^2 = 4 \cdot 3(4m^2 + 1) \rightarrow 64m^2 = 48m^2 + 12 \rightarrow m = \pm\frac{\sqrt{3}}{4}$$

$$\rightarrow (8m)^2 = 4 \cdot 3(4m^2 + 1) \rightarrow 64m^2 = 48m^2 + 12 \rightarrow m = \pm\frac{\sqrt{3}}{2}$$

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6) As shown in the diagram, right triangle ABC, with right angle C, is inscribed in equilateral triangle PQR. If PC = 3, BP = CQ = 2, compute AQ.

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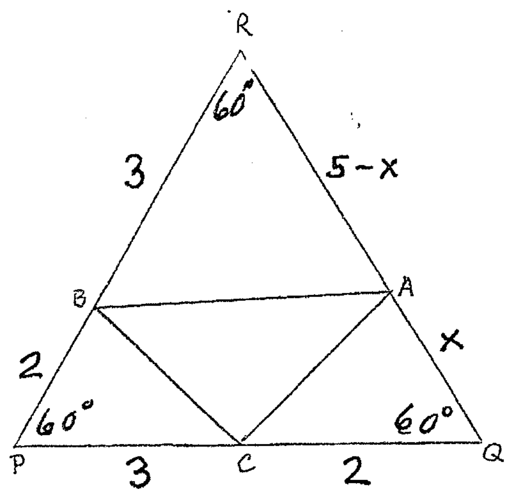


Use the Law of Cosines to find the sides of $\Delta ABC$.

$$BC^2 = 2^2 + 3^2 - 2 \cdot 2 \cdot 3 \cdot \cos 60 = 4 + 9 - 6 = 7$$

$$AC^2 = 2^2 + x^2 - 2 \cdot 2 \cdot x \cdot \cos 60 = 4 + x^2 - 2x$$

$$AB^2 = 3^2 + (x-5)^2 - 2 \cdot 3 \cdot (5-x)\cos 60$$

$$\rightarrow AB^2 = 9 + x^2 - 10x + 25 - 15 + 3x$$

$$\rightarrow AB^2 = x^2 - 7x + 19$$

Finally, the right triangle $AB^2 = BC^2 + AC^2$

$$x^2 - 7x + 19 = x^2 - 2x + 4 + 7$$

$$-5x = -8 \rightarrow x = \frac{8}{5}$$