CSAML
March 31, 2014

Possible solutions

Round I: Arithmetic and Number Theory

1. How many palindromic numbers can be formed by putting any arrangement of digits from 0 to 9 into the five blank spaces of 2 __ __ 8 8 __ __ 4 __?

(answer: 10) The last space must have a 2, the 1st blank space must be a 4, and the 3rd blank space must be an 8. We now have 2 4 _ 8 8 8 _ 4 2. This leaves two spaces that can have any digit from 0 to 9, or 10 different digits.

2. Determine the value of *a* in the decimal number 62894*a*44. The entire number is divisible by 48.

(answer: 5) $48 = 2^4(3)$,For 3 to be a factor, the sum of the digits must be a multiple of 3, $\Sigma = 37 + a$, so a = 2, 5, or 8. For the given number to be divisible by 2^4, 4a44 must be divisible by 2^4. Try each of the three vales of a, and a = 5.

3.The sales tax on an item is r%, where r is an integer, $0 < r < 50$. Let x be the price of an article in pennies, with $0 < x < 200$. If the article costs precisely 2 dollars when unrounded sales tax(in pennies) is added, what is the cost in dollars of the item before the sales tax?

(answer: \$1.60 or 1.6) $x\left(1+\frac{r}{100}\right) = 200 \to x(100+r) = 20000 \to x = \frac{20000}{100+r} < 200$

$20000 = 2^5(5^4)$, the factor has to be $\begin{array}{l} 100+r < 150 \to 5^3 = 125 \to r = 25 \\ 2^5(5) = 160, r = 25\%, x = 1.6 \end{array}$

Round II: Algebra I (Real numbers and no transcendental functions)

1. Solve for real *x*. $x - 1 + 2\cdot 3 = 4[(5x-6)+7(8-9x)]$

(answer: $\frac{195}{233}$) $x + 5 = 4[5x - 6 + 56 - 63x] \to x + 5 = 200 - 232x \to x = \frac{195}{233}$

2. Given: $f(x) = 5x + a,\ h(x) = 2x - b,$ and $f(h(x)) = h(f(x))$ Determine: $\frac{a}{b}$.

(answer: -4)

$$f(2x-b) = 5(2x-b) + a \text{ and } h(5x+a) = 5(5x+a) - b$$

$$10x - 5b + a = 10x + 2a - b \to a = -4b \to \frac{a}{b} = -4$$

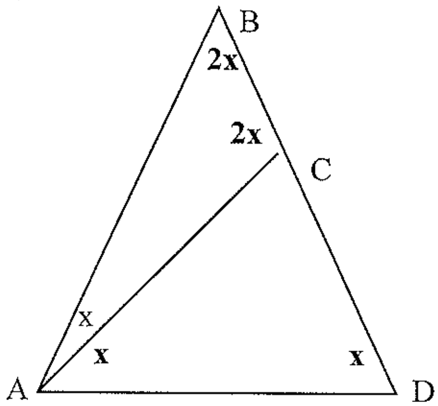
3. Solve for x: . $x + \frac{23}{5}\sqrt{x} = 2$

(answer, $\frac{4}{25}$) $x + \frac{23}{5}\sqrt{x} = 2 \to 5x + 23\sqrt{x} - 10 = 0 \to (5\sqrt{x}-2)(\sqrt{x}+5) = 0 \to \sqrt{x} = \frac{2}{5} \to x = \frac{4}{25}$

Round III: Geometry

1. In triangle ABD, AB = AC = CD and AD = BD. Find $m\angle ADC$ (in degrees).

(answer: 36°)



In triangle ABC the sum of the angles is 5x = 180, so x = 36°.

2. Quadrilateral QRST is inscribed in a circle. Given that $m\angle QTS = x^2 + 2x + 115$ degrees and $m\angle QRS = 3x + 71$ degrees, find all possible values for the measure of $\angle QTS$ in degrees.

(answer: 115°, 118°)

$$m\angle R = \frac{1}{2} m\overset{\frown}{QTS} \text{ and } m\angle T = \frac{1}{2} m\overset{\frown}{QRS}$$

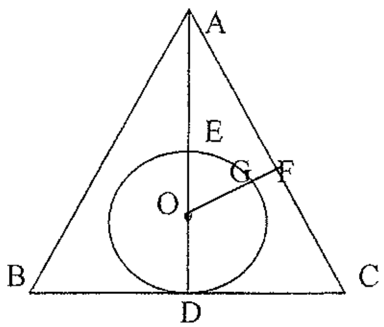
$$m\overset{\frown}{QTS} + m\overset{\frown}{QRS} = 360° \rightarrow m\angle R + m\angle T = \frac{1}{2}\left(m\overset{\frown}{QTS} + m\overset{\frown}{QRS}\right) = 180°$$

$$x^2 + 2x + 115 + 3x + 71 = 180 \rightarrow x^2 + 5x + 6 = 0 \rightarrow x = -2, -3$$

$$m\overset{\frown}{QTS} = x^2 + 2x + 115 = 115°, 118°$$

3. ABC is an equilateral triangle of side 2. $\overline{AD}$ is an altitude of ΔABC, circle O is tangent to side $\overline{BC}$ at D and AD = 2AE. Determine the shortest distance from a point on circle O to $\overline{AC}$.

(answer: $\frac{\sqrt{3}}{8}$)



E bisects $\overline{AD}$, so AD = 4r. Since ΔABC is equilateral with side equal to 2, $4r = \sqrt{3}$ and then $r = \frac{\sqrt{3}}{4}$. ΔACD is 30-60-90 with $m\angle D = 90$, so

AO = 3r, $OF = \frac{1}{2}(3r) = \frac{3r}{2}, GF = \frac{3r}{2} - r = \frac{r}{2}$

$$GF = \frac{\sqrt{3}}{8}$$

Round IV:Algebra II Including logarithms and exponential functions.

1.Factor completely: $x^4 - 4x^3 + 4x^2 - 9y^2$.

(answer: $(x^2 - 2x + 3y)(x^2 - 2x - 3y)$)

$$x^4 - 4x^2 + 4x^2 - 9y^2 \rightarrow (x^2 - 2x)^2 - 9y^2$$
$$(x^2 - 2x + 3y)(x^2 - 2x - 3y)$$

2. Compute: $\sqrt{11 + \sqrt{72}} + \sqrt{11 - \sqrt{72}}$.

(answer: 6) Let $x = \sqrt{11 + \sqrt{72}} + \sqrt{11 - \sqrt{72}}$

$$x^2 = 11 + \sqrt{72} + 2\sqrt{(11 + \sqrt{72})(11 - \sqrt{72})} + 11 - \sqrt{72}$$
$$x^2 = 22 + 2\sqrt{121 - 72} \rightarrow x^2 = 22 + 2\sqrt{49} = 36$$
$$x = 6$$

3. If $f(x) = 1 + x + x^2$, $g(x) = 2 + 3x + x^2$, and $h(x) = 5 - x + 2x^2$ such that for all real values of x: $a(f(x)) + b(g(x)) + c(h(x)) = 2 - 8x + 3x^2$. Express your solution as an ordered triple (a, b, c)

(answer, (5, -4, 1)

$$a(1 + x + x^2) + b(2 - 3x + x^2) + c(5 - x + 2x^2) = 2 - 8x + 3x^2$$

$$\rightarrow \begin{cases} a + 2b + 5c = 2 \\ a + 3b - c = -8 \\ a + b + 2c = 3 \end{cases} \rightarrow \begin{cases} b - 6c = -10 \\ 2b - 3c = -11 \end{cases} \rightarrow c = 1, b = -4, a = 5 \rightarrow (5, -4, 1)$$

Round V:Analytic Geometry

1. What is the area of the region enclosed by the graph of $|x - 1| + |y + 1| = 2$?

(Answer: 8) The figure is a square with vertices at (1, 1), (3, -1), (1, -3) and (-1, -1). You can use the formula for the area of a rhombus $A = \frac{1}{2}d_1d_2 = \frac{1}{2}(4)(4) = 8$.

2. Circle $(x-1)^2 + (y-2)^2 = 4$ passes through the focal points of an ellipse whose major axis is parallel to the x-axis. The circle is also internally tangent to the ellipse. Determine the largest y-intercept of the ellipse.(answer: $\frac{4+\sqrt{14}}{2}$) The circle has its center at (1, 2) and its radius is 2. Now the ellipse is of the form: $\frac{(x-1)^2}{a^2} + \frac{(y-2)^2}{4} = 1$ and $a^2 = b^2 + c^2 = 4 + 2^2 = 8$. Now the ellipse is $\frac{(x-1)^2}{8} + \frac{(y-2)^2}{4} = 1$. Now find the largest y intercept, x = 0.

$$\frac{1}{8} + \frac{(y-2)^2}{4} = 1 \rightarrow (y-2)^2 = 4 - \frac{1}{2} = \frac{7}{2}$$

$$\rightarrow y-2 = \sqrt{\frac{7}{2}} \rightarrow y = \frac{\sqrt{14}+4}{2}$$

3. The parabola with equation $y = ax^2 + bx + c$ and vertex (h, k) is reflected about the line $y = k$. This results in a parabola with equation $y = dx^2 + ex + f$. What is the value of $a + b + c + d + e + f$?

(answer: 2K) The equations of the two parabolas with vertex (h, k) that reflect about y = k are $(x-h)^2 = p(y-k)$ and $(x-h)^2 = -p(y-k)$. Multiply these out and solve for y and you get $(x-h)^2 = p(y-k)$ and $(x-h)^2 = -p(y-k)$, so $y = \frac{x^2 - 2xh + h^2 + pk}{p}$ and $y = \frac{x^2 - 2xh + h^2 - pk}{-p}$.

Now find a + b + c + d + f.

$a = \frac{1}{p}$, $b = \frac{-2h}{p}$, $c = \frac{h^2 + pk}{p}$, $d = \frac{1}{-p}$, $e = \frac{-2h}{-p}$, $f = \frac{h^2 - pk}{-p}$. Notice a = -d, b = -e, so we need c + f.

$$c + f = \frac{h^2+pk}{p} + \frac{h^2-pk}{-p} \rightarrow \frac{h^2+pk}{p} - \frac{h^2-pk}{p} = \frac{2kp}{p} = 2k.$$

Round VI: Trigonometry, Complex Numbers

1. Simplify: $\frac{i^{-5} - i^{24}}{i^{-7} + i}$.

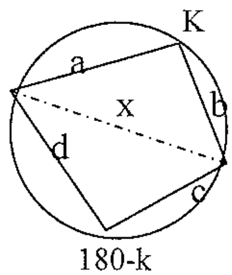
(answer: $-\frac{1}{2} + \frac{1}{2}i$) $\frac{i^{-5} - i^{24}}{i^{-7} + i} = \frac{i^7}{i^7}\frac{i^{-5} - i^{24}}{i^{-7} + i} = \frac{i^2 - i^{31}}{1 + i^8} = \frac{-1-(-i)}{1+1} = \frac{-1+i}{2} = \frac{-1}{2} + \frac{i}{2}$

2. Compute the least positive degree measure for x for which $8\sin x\cos^5 x - 8\sin^5 x\cos x = \sqrt{2}$.

(answer: $\frac{45}{4}$)

$$8\sin x\cos x(\cos^4 x-\sin^4 x)=\sqrt{2}$$
$$4\sin 2x(\cos^2 x+\sin^2 x)(\cos^2 x-\sin^2 x)=\sqrt{2}$$
$$4\sin 2x\cos 2x=\sqrt{2}\rightarrow 2\sin 4x=\sqrt{2}\rightarrow \sin 4x=\frac{\sqrt{2}}{2}$$
$$4x=45\rightarrow x=\frac{45}{4}$$

3. In a quadrilateral inscribed in a circle the sides are of length *a, b, c, d* in that order. Angle *K* is the angle between the two sides of length *a* and *b*. Find an algebraic formula for the cosine of angle *K* in terms of *a, b, c,* and *d*.



Since $\cos(180-K)=-\cos K$

$$x^2=a^2+b^2-2ab\cos K=c^2+d^2+2cd\cos K$$
$$a^2+b^2-c^2-d^2=(2ab+2cd)\cos K$$
$$\cos K=\frac{a^2+b^2-c^2-d^2}{2ab+2cd}$$

Team Round

1) In the mini-Sudoku puzzle shown, each row of 4, each column of 4, and each of the 2 by 2 boxes must contain all the numbers 1, 2, 3, 4. The puzzle doesn't have enough information for a unique solution. Find the sum of all possible entries into the box labeled "x" that are part of a proper solution.

1			
			2
		3	
	x		

(answer: 8) The box beside the 1 has to be a 2 and the box below the 3 has to be a 2. So x cannot be 2. It can be all others 1, 3, 4. The sum is 8.

2) John ran entire race in 50 minutes. The race was comprised of 3 distinct laps of equal length. He ran first lap at an average speed of 12 km/hr. He ran each of the last two laps at an average speed of 16 km/hr. How long in kilometers was the whole course. (total distance)?

(answer: 12 km) Since the time intervals are given use $T=\frac{D}{R}$.

Let x = the length of one lap. $\frac{x}{12}+\frac{x}{16}+\frac{x}{16}=\frac{5}{6}\rightarrow 4x+3x+3x=40$

$$10x=40\rightarrow x=4$$

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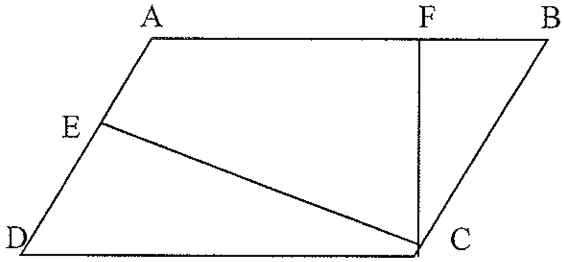
3 laps = 12 km.

3) In parallelogram ABCD, $\overline{CF} \perp \overline{AB}$ and $\overline{CE} \perp \overline{AD}$. If CF = 2, CE = 4 and FB is one-sixth of AB, what fraction of the area of parallelogram ABCD is the area of quadrilateral AFCE?



(answer: $\frac{7}{12}$) Let AB = 6x, now CD =6x and FB = x. ΔDEC ≈ ΔBFC and

$\frac{CE}{CF} = \frac{4}{2} \to \frac{DE}{BF} = \frac{2x}{x} \to DE = 2x$. The area of ABCD is 12 x and

the area of AFCE =12x - ΔFBD -ΔEDC.

$$12x - \Delta FBC - \Delta EDC \to 12x - \frac{1}{2}(x)(2) - \frac{1}{2}(2x)(4) = 7x$$

$$\frac{\text{Area AFCE}}{\text{Area ABCD}} = \frac{7x}{12x} = \frac{7}{12}$$

4) Solve for x: $2 = \sqrt{x + \sqrt{2x-1}} + \sqrt{x - \sqrt{2x-1}}$.

(answer: $\frac{3}{2}$)

$$2 = \sqrt{x + \sqrt{2x-1}} + \sqrt{x - \sqrt{2x-1}}$$

$$(2)^2 = \left(\sqrt{x + \sqrt{2x-1}} + \sqrt{x - \sqrt{2x-1}}\right)^2$$

$$4 = x + \sqrt{2x-1} + 2\sqrt{\left(x + \sqrt{2x-1}\right)\left(x - \sqrt{2x-1}\right)} + x - \sqrt{2x-1}$$

$$4 = 2x + 2\sqrt{x^2 - (2x-1)} \to 4 = 2x + 2(x-1)$$

$$6 = 4x \to x = \frac{3}{2}$$

5) A circle with center on the y-axis passes through the points (-7, -6) and (20, 3). The circle intersects the positive x-axis at (α, 0). Find α.

(answer: $\sqrt{301}$)

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The circle begins with $x^2 + (y-h)^2 = r^2$. Substitute each point (-7, -6) and (20, 3) into this equation

$$(-7)^2 + (-6-h)^2 = (20)^2 + (3-h)^2$$

and set the equations equal to each other: $49 + 36 + 12h + h^2 = 400 + 9 - 6h + h^2$

$$18h = 324 \rightarrow h = 18$$

Now find the radius: $x^2 + (y-18)^2 = r^2 \rightarrow 400 + 225 = 625 \rightarrow r = 25$

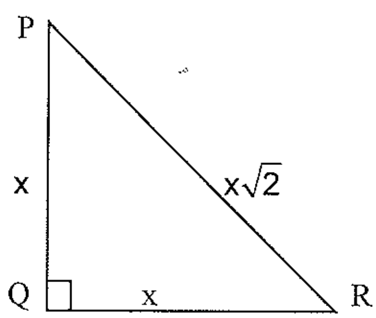
Find the point (a , 0)

$$(a-0)^2 + (0-18)^2 = 625$$

$$a^2 = 625 - 324 = 301 \rightarrow a = \sqrt{301}$$

6) Given right triangle PQR, right angle Q. $\frac{PQ}{QR} = \frac{\csc 78^\circ}{\sec 12^\circ}$ and $PR = \frac{1+\sqrt{2}}{5}$. Determine the area of ΔPQR.

(answer: $\frac{3+2\sqrt{2}}{100}$)



Since sec 12° = csc 78°, it follows that the triangle is a right isosceles triangle.

$$PR = \frac{1+\sqrt{2}}{5} = x\sqrt{2} \rightarrow x = \frac{1+\sqrt{2}}{5\sqrt{2}}$$

Now the area $= \frac{1}{2}x^2 = \frac{1}{2}\left(\frac{1+\sqrt{2}}{5\sqrt{2}}\right)^2 = \frac{3+2\sqrt{2}}{100}$