CSAML 2016 Solutions

April 6, 2016

Round I: Arithmetic & Number Theory

1. (1 point) The mean score on a test taken by 21 students on one day was 82. Three students were absent on the day of test and took it the next day. These 3 students did so well that the mean score for all 24 students was raised 1 (one) point. What was the mean score for these 3 absent students?

$$21(82)+3x=24(83)$$
$$3x=1992-1722$$
$$3x=270\rightarrow x=90$$

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2. (2 points) If $\frac{1}{a+\sqrt{b}}=a-\sqrt{b}$ with a, b are positive integers and $a<10$, determine how many ordered pairs (a, b) exist, such that a + b is prime.

Notice that $a^2-b=1$

So, if $a=1$, the b = 0, and $a+$b = 1 (not a prime number)

a	b	a + b	Prime?		a	b	a+b	Prime
2	3	5	Yes		6	35	41	Yes
3	8	11	Yes		7	48	55	No
4	15	19	Yes		8	63	71	Yes
5	24	29	Yes		9	80	89	Yes

There are 7 ordered pairs.

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3. (3 points) A and B are integers greater than 1 such that $A^9B^3=(57)^3(117)^3$. Find $A+B$.

Notice that 57 is a multiple of 3, the sum of the digits is a multiple of 3 and that 117 is a multiple of 9, the sum of its digits is 9.

So,

$$57^3\cdot 117^3\rightarrow 3^3\cdot 19^3\cdot\left(3^2\right)^3\cdot 13^3\rightarrow 3^9\cdot(19\cdot 13)^3$$
$$A=3, B=19\cdot 13=247\rightarrow A+B=250$$

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Round II: Algebra I (Real numbers and no transcendental functions)

1. (1 point) Find all values of a such that $|a-4|^2 - |a-4| = 6$.

$y^2 - y - 6 = 0 \rightarrow (y-3)(y+2) = 0 \rightarrow y = 3, y = -2$

$|a-4| = 3$ or $|a-4| \neq -2 \rightarrow a - 4 = 3$ or $a - 4 = -3$

$a = 7$ or $a = 1$

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2. (2 points) Evaluate: $x^3 - x^2y - xy^2 + y^3$ if $x = 2015$ and $y = 2016$.

$x^3 - x^2y - xy^2 + y^3 \rightarrow x^2(x-y) - y^2(x-y)$

$(x^2 - y^2)(x-y) \rightarrow (x+y)(x-y)(x-y)$

$\rightarrow (2015+2016)(2015-2016)(2015-2016)$

$\rightarrow 4031(-1)(-1) \rightarrow 4031$

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3. (3 points) When $kx^2 + 5x + 6$ is divided by $x+2$, the remainder is the same as when $kx^2 + 5x + 6$ is divided by $x-3$. Find all possible values of k.

Use synthetic division:

$\underline{-2}|\ k + 5 + 6$	$\underline{3}|\ k + 5 \quad + 6$	$4k - 4 = 21 + 9k \rightarrow 5k = -25$
-2k+4k-10	3k+15+9k	$k = -5$
k-2k+5+$\lfloor 4k-4$	k+5+3k+$\lfloor$21+9k	

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Round III: Geometry (figures are not necessarily drawn to scale)

1. (1 point) How many circular pipes with an inside diameter of one inch would be needed to carry the same amount of water as a single circular pipe with an inside diameter of one foot?

$$x\not{\pi}\left(\frac{1}{2}\right)^2 = \not{\pi}(6)^2$$

$$\frac{x}{4} = 36 \rightarrow x = 144$$

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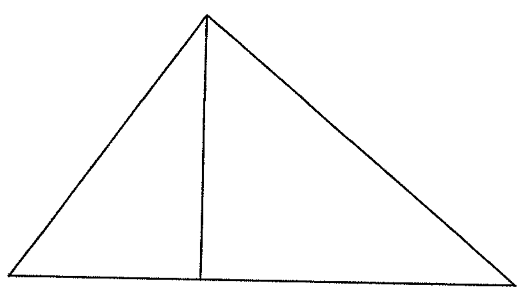
2. (2 points) Triangle ABC, shown below, has a right angle at B. Segment BD is an altitude of the triangle. AB = 4 and CD = 6. Compute x + y + z.



$$\frac{x}{4} = \frac{4}{x+6} \rightarrow x^2 + 6x = 16 \rightarrow (x+8)(x-2) = 0$$

$$x \neq -8 \text{ or } x = 2$$

$$\frac{2}{y} = \frac{y}{6} \rightarrow y^2 = 12 \rightarrow y \neq -2\sqrt{3} \text{ or } y = 2\sqrt{3}$$

$$\frac{6}{z} = \frac{z}{8} \rightarrow z^2 = 48 \rightarrow z \neq -4\sqrt{3} \text{ or } z = 4\sqrt{3}$$

$$x + y + z = 2 + \sqrt{6}$$

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3. (3 points) A circle is inscribed in a 60° sector of a larger circle. The smaller circle is internally tangent to the larger circle, and is tangent to the two radii that form the sector. What is the ratio of the radius of the smaller circle to the radius of larger circle?

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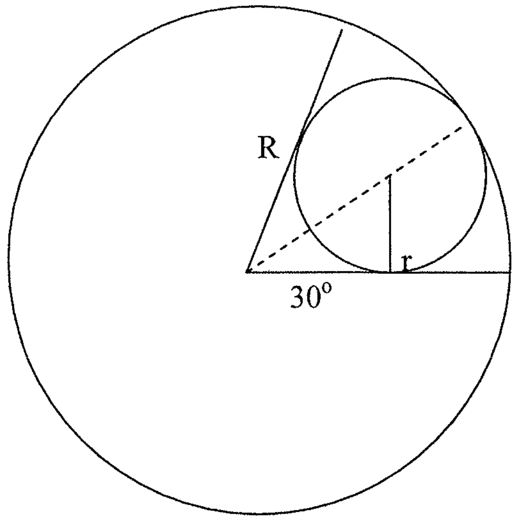

R – r = 2r (in a 30°-60°-90° triangle

$R = 3r \rightarrow \frac{r}{R} = \frac{1}{3}$

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Round IV: Algebra II

1. (1 point) A function f is given by $f(x) = \frac{4x+3}{x}$. Find the value of a such that $f(2a) = 2f(a)$.

$$\frac{8a+3}{2a} = 2\left(\frac{4a+3}{a}\right) \rightarrow 4a(4a+3) = a(8a+3)$$

$$16a^2 + 12a = 8a^2 + 3a \rightarrow 8a^2 + 9a = 0 \rightarrow a \neq 0, a = \frac{-9}{8}$$

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2. (2 points) The graph of the function f is shown below. How many solutions does the equation $f(f(x)) = 6$ have?

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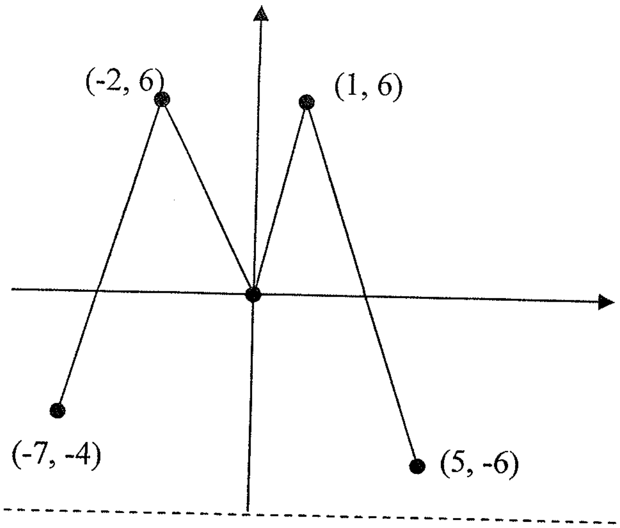

If $f(f(x)) = 6$ then $f(x) = -2$ or $f(x) = 1$. The horizontal line $y = -2$ intersects f twice, so $f(x) = -2$ has 2 solutions. Likewise $f(x) = 1$ has 4 solutions. Thus, there are 6 solutions for $f(f(x)) = 6$.

3. (3 points)

Find, in terms of b, the remainder when $x^3 + bx + b$ is divided by $x - b + 1$.

$$P(x) = x^3 + bx + b \rightarrow P(b-1) = (b-1)^3 + b(b-1) + b$$

$$\rightarrow b^3 - 3b^2 + 3b - 1 + b^2 - \cancel{b} + \cancel{b} \rightarrow= b^3 - 2b^2 + 3b - 1$$

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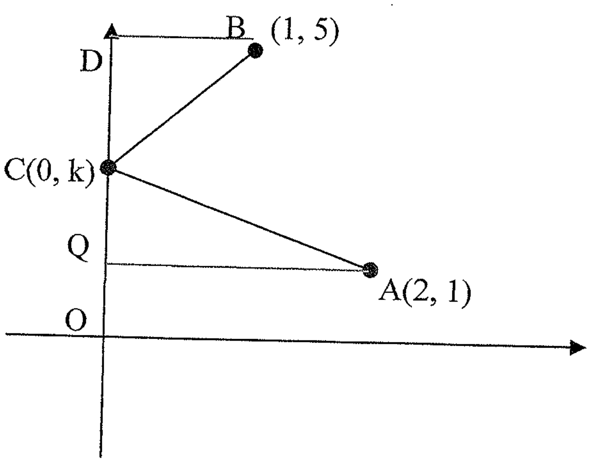
Round V: Analytic Geometry

1. (1 point) Refer to the diagram below to Find k so that $\angle DCB \cong \angle ACO$.



Draw BD and AQ perpendicular to the y-axis.
Let CD = 5 – k and CQ = k – 1.
Now you have 2 similar triangles, BDC and AQC. Set up a proportion:

$$\frac{2}{1} = \frac{k-1}{5-k}$$

$$k-1=10-2k$$

$$3k = 11 \rightarrow k = \frac{11}{3}$$

2. (2 points) Find the distance between the vertex of $f(x) = 2x^2 + 16x + 35$ and the vertex of its inverse relation.

Factor and complete the square to get a parabola in standard form.

$$y = 2\left(x^2 + 8x + 16\right) + 35 - 32$$

$$y = 2\left(x + 4\right)^2 + 3$$

So the two vertices are

$$V_f = (-4,3) \text{ and } V_n = (3,-4) \rightarrow d = \sqrt{(-4-3)^2 + (3+4)^2}$$

$$d = 7\sqrt{2}$$

3. (3 points) Find the equation of the common chord of the two circles

$$(x+1)^2 + (y-3)^2 = 7 \text{ and } \left(x - \frac{3}{2}\right)^2 + (y+4)^2 = \frac{169}{4}.$$

Leave your answer in the form Ax + By + C = 0, where A, B, C are integers with no common factor other than 1, and A > 0.

$$(x+1)^2 + (y-3)^2 = 7 \rightarrow x^2 + 2x + 1 + y^2 - 6y + 9 = 7$$

$$\left(x - \frac{3}{2}\right)^2 + (y+4)^2 = \frac{169}{4} \rightarrow x^2 - 3x + \frac{9}{4} + y^2 + 8y + 16 - \frac{169}{4}$$

$$\begin{cases} x^2 + 2x + 1 + y^2 - 6y + 9 = 7 \\ x^2 - 3x + \frac{9}{4} + y^2 + 8y + 16 = \frac{169}{4} \end{cases} \rightarrow -5x + \frac{5}{4} + 14y + 7 = \frac{141}{4}$$

$$\rightarrow -5x + 14y = 27 \rightarrow 5x - 14y + 27 = 0$$

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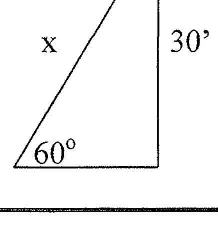
Round VI: Trigonometry

1. (1 point) An ant is standing on horizontal ground. When the ant looks at the top of a building, the angle of elevation is 60°. The height of the building is 30 meters. How far, in meters, is the top of the building from the ant?



This is a 30-60-90 right triangle with sides in the proportion $1 : 2 : \sqrt{3}$. In this triangle we have $\frac{x}{2} = \frac{30}{\sqrt{3}} \rightarrow x = \frac{60}{\sqrt{3}} = 20\sqrt{3}$

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2. (2 points) Find the value of $\tan x$ if $\dfrac{\cos(x-60^o)}{\sin x}=2$.

$$\frac{\cos(x-60)}{\sin x}=2 \rightarrow \frac{\cos x \sin 60+\sin x \sin 60}{\sin x}=2$$

$$\frac{1}{2}\cos x+\frac{\sqrt{3}}{2}\sin x=2\sin x \rightarrow \frac{1}{2}\cos x=\left(2-\frac{\sqrt{3}}{2}\right)\sin x$$

$$\tan x=\frac{\frac{1}{2}}{\left(2-\frac{\sqrt{3}}{2}\right)} \rightarrow \frac{1}{4-\sqrt{3}}\cdot\frac{4+\sqrt{3}}{4+\sqrt{3}}=\frac{4+\sqrt{3}}{13}$$

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3. (3 points) The three solutions of the equation $x^3=8i$ are $a_1+b_1i,\ a_2+b_2i,\ a_3+b_3i$, where a_1,a_2,a_3,b_1,b_2,b_3 are real. Find $b_1^2+b_2^2+b_3^2$.

$$8i=8(0+i) \rightarrow r=8, \theta=\frac{\pi}{2}$$

$$x^3=8i \rightarrow x=2cis\left(\frac{\frac{\pi}{2}+2\pi n}{3}\right);n=0,1,2$$

$$x_1=\sqrt{3}+i, x_2=\sqrt{3}+i, x_3=0-2$$

$$b_1=1, b_2=1, b_3=-2 \rightarrow b_1^2+b_2^2+b_3^2=1+1+4=6$$

Team Round

April 6, 2016

1. On the planet Mathlandia there are as many days in a week as there are weeks in a month. The number months in a year is half the number of days in a week. If there are 864 days in a Mathlandia year how many days are in a Mathlandia week? (Note: The number of months in the year is an integer, and the months are of equal length.)

Let x = days per week

x = weeks per month

x/2 = months per year

days per year $\dfrac{x^3}{2}=864 \rightarrow x^3=1728 \rightarrow x=12$

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2. Solve for x: $\dfrac{2-\dfrac{4}{x}}{\left(1+\dfrac{3}{x}\right)\left(1-\dfrac{5}{x}\right)}=\dfrac{1-\dfrac{8}{x}}{1-\dfrac{5}{x}}+1.$

$\rightarrow \dfrac{2x-4}{x}\cdot\dfrac{x^2}{(x+3)(x-5)}=\dfrac{x-8+x-5}{x-5}\rightarrow 2x^2-4x=(2x-13)(x+3)$

$\rightarrow 2x^2-4x=2x^2-7x-39\rightarrow 3x=-39\rightarrow x=-13$

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3. Given:

$m\angle D=40$

$\overrightarrow{AF}$ bisects $\angle CAB$

$\overrightarrow{BD}$ bisects $\angle CBE$

Determine: $m\angle C$

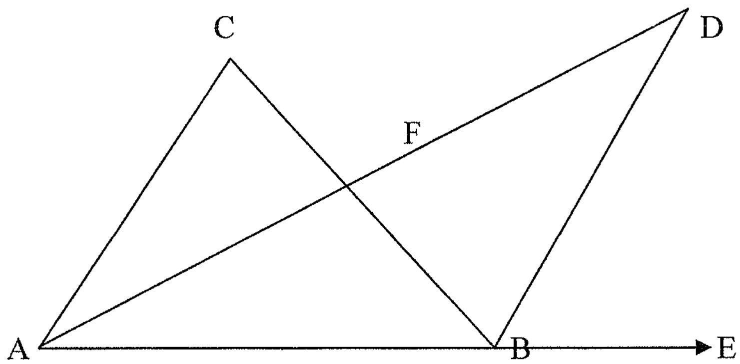


Diagram below:

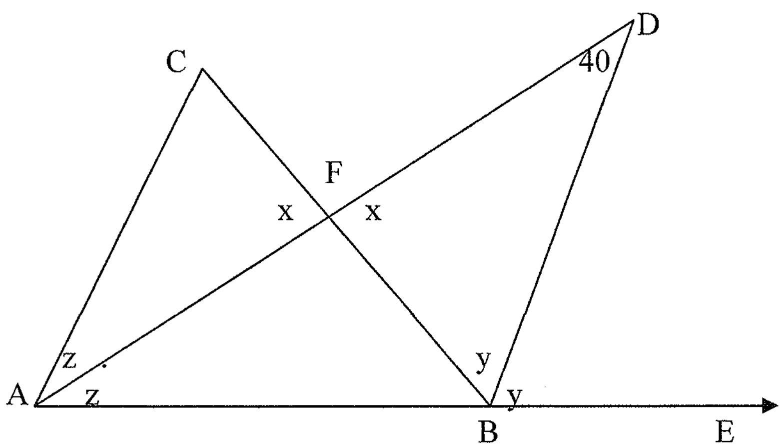
By the exterior angle rule: $2y=2z+C\rightarrow C=2y-2z$ $2y = 2z + C \rightarrow C = 2y - 2z$

Sum of the angles in a triangle: $x+y+40=180\rightarrow x+y=140$ $x + y + 40 = 180 \rightarrow x + y = 140$

Sum of the angles in a triangle: $z+x+C=180$ $z + x + C = 180$

Substitute for C : $x+y+2y-2z=180$ $x + y + 2y - 2z = 180$

$\begin{cases} x+2y-z=180 \\ x+y=140 \end{cases}\rightarrow y-z=40\rightarrow C=2y-2z=80$



Or: By the exterior angle for ΔABD: $y=z+40^o\rightarrow y-z=40^o$

By the exterior angle for

$\Delta ABC: 2y=2z+C\rightarrow C=2y-2z\rightarrow 2(y-z)=2\cdot 40=80$

4. Solve the following system of equations for x and y.

$$\begin{cases}\log_2(x+1)-\log_2 y=1\\ \log_3(x+y)=\log_3 x+\log_3 y\end{cases}$$

$$\begin{cases}\log_2(x+1)-\log_2 y=1\\ \log_3(x+y)=\log_3 x+\log_3 y\end{cases} \to \begin{cases}\dfrac{x+y}{2}=1\\ x+y=xy\end{cases} \to \begin{cases}x=2y-1\\ x+y=xy\end{cases} \to$$

$$2y-1+y=y(2y-1)$$

$$3y-1=2y^2-y \to 2y^2-4y+1=0$$

$$y=\frac{-4\pm\sqrt{4^2-4(2)(1)}}{2(2)}=\frac{2\pm\sqrt{2}}{2}$$

You can use only the positive value of y, so $y=\frac{2+\sqrt{2}}{2}$ and

$$x=2\left(\frac{2+\sqrt{2}}{2}\right)-1=1+\sqrt{2}$$

$$x=1+\sqrt{2},\ y=\frac{2+\sqrt{2}}{2}$$

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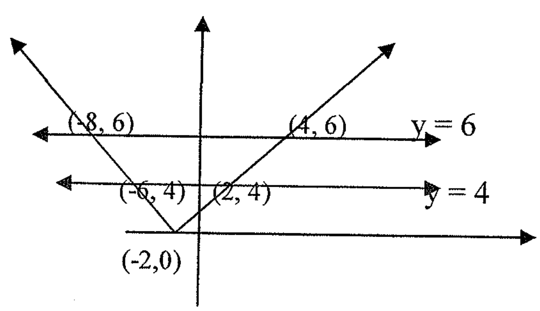
5. Find the area of the plane region formed by the intersection of the graphs

$$\begin{cases}y\geq|x+2|\\ |y-5|\leq 1\end{cases}.$$

You can draw each graph and then find the area.



The area between the three line is a trapezoid or you can do two triangles:

$$\frac{1}{2}(12)(6)-\frac{1}{2}(8)(4)=36-16=20$$

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6. Solve, for $0 < x < \pi$, the equation $\csc x - \cot x = \sqrt{3}$.

GIVE YOUR SOLUTION(S) IN RADIANS.

$\csc x - \cot x = \sqrt{3}$

$(\csc x - \cot x)^2 = 3 \rightarrow \csc^2 x - 2\csc x \cot x + \cot^2 x = 3$

$\frac{1}{\sin^2 x} - \frac{2\cos x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = 3 \rightarrow (1-\cos x)^2 = 3\left(1-\cos^2 x\right)$

$(1-\cos x) = 3\,(1+\cos x) \rightarrow -2 = 4\cos x \rightarrow \cos x = \frac{-1}{2} \rightarrow x = \frac{2\pi}{3}$