NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

PLAYOFFS – 2015

Round 1: Arithmetic and Number Theory

1. ______________

2. ______________

3. ______________

1. For integers x and y, $x * y = x^y + y^x$. If $2 * A = 100$, compute A.

2. Given a and b are natural numbers and $a^2 + b^3 < 100$, how many ordered pairs satisfy this inequality?

3. Each square of a 3 x 3 grid is filled with a distinct integer from 1 to 9, inclusive.

   Compute the probability that the randomly-filled grid at the right will have each of a, b, and c less than 5, and each of p, q, and r greater than 5.

a	b	c
x	5	y
p	q	r

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Round 2: Algebra 1

1. ______________________

2. ______________________

3. (__________ , __________)

1. If $312_{(x-1)} = 211_{(x+1)}$, find the value of x.

2. The square of the reciprocal of 3 less than a number is $\frac{25}{144}$. Compute two possible values of the number.

3. A and B denote the largest pair of values in the following list, where $A > B$:

$$1^{160},\ 2^{140},\ 3^{120},\ 4^{100},\ 5^{80},\ 6^{60},\ 7^{40},\ 8^{20},\ 9^{0}$$

Compute the ordered pair (A, B). Leave the answers in exponential form as shown.

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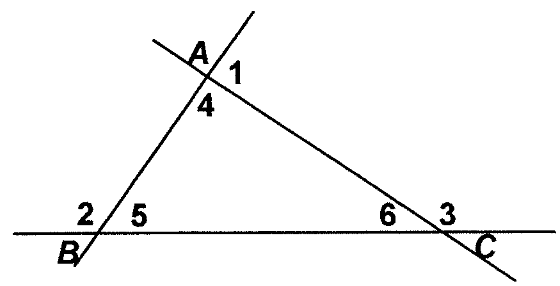
Round 3: Geometry

1. ______:______:______

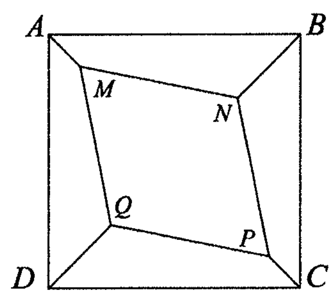
2. ____________________

3. ____________________

1. The exterior angles of ΔABC satisfy the following condition:
$m\angle 1 : m\angle 2 : m\angle 3 = 7:10:13$.
Compute the ratio $m\angle 4 : m\angle 5 : m\angle 6$.



2. $ABCD$ is a square with a side length of 16. From A and C segments of length 2 are drawn into the square at angles of 45 with the sides. From B and D segments of length 4 are drawn in a similar fashion. Compute the area of $MNPQ$.



3. Given: An isosceles triangle where the legs each have length a and the base has length c. Let R denote the radius of the circumscribed circle. Determine a simplified expression for R^2 in terms of a and c

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Round 4: Algebra 2

1. _______________

2. _______________

3. _______________

1. If $S_n = i^n + i^{-n}$, where n is any integer, and $i = \sqrt{-1}$, compute <u>all</u> possible values of S_n.

2. The roots of $\left(\frac{\log 0.\overline{3}}{\log 81}\right)x^2 + \left(\frac{\log 8}{\log 4}\right)x + \log 1000 = c$ are equal. Each log expression is understood to be a common log, i.e. $\log_{10}$. Compute c.

3. In the expansion of $\left(1+x^2\right)^t$ for a <u>positive</u> integer t, the coefficient of x^8 is six times the coefficient of x^4. If the first term in the expansion is 1, compute the ninth term in the expansion.

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Round 5: Analytic Geometry

1. (____, ____, ____, ____)

2. ____________

3. (________ , ________)

1. Given the equation of a circle: $x^2 - 8x + y^2 + 14y + 40 = 0$. When written in simplified form, the area between a chord and an arc cut off by a central angle of 45° can be expressed as $\frac{a}{b}\left(\pi - c\sqrt{d}\right)$, where a, b, c, and d are integers. Determine (a, b, c, d)

2. Given the equation $y^2 - 8y + 13 = x$, compute the number of square units in the area of the trapezoid whose vertices are on the parabola and having x-coordinates which are 4 more and 16 more than the x-coordinate of the vertex.

3. Consider an ellipse with center $(0, 0)$ and the major axis on the y-axis. One vertex is $\left(0, -2\sqrt{10}\right)$ and one point on the ellipse is $P(-3, -4)$. A line L_1 through P has x-intercept 1 and intersects a line L_2, which contains a focus $F(0, f)$, $f > 0$, and is perpendicular to L_1. The intersection of L_1 and L_2 is $Q(m, n)$.
Determine the ordered pair (m, n).

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Round 6: Trig and Complex Numbers

1. (________ , ________)

2. ____________________

3. (_____, _____, _____, _____)

1. The following can be expressed in $a+bi$ form. Compute the ordered pair (a,b).

$$\left(\sqrt{2}cis\frac{7\pi}{24}\right)^4\left(\sqrt{3}cis\frac{5\pi}{18}\right)^6$$

2. Given $\cos(a-180°)=-0.8$, $\sin(b+90°)=0.6$, $0\le a<90°$, and $0\le b<90°$, compute $\cos(a+b)-\sin(a-b)$.

3. There are 4 solutions to the equation $\sin(x+17)=\cos(2x-23)$ over the interval $0\le x<360°$. If the solutions are denoted A, B, C and D, where $A<B<C<D$, compute (A,B,C,D).

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Team Round - Place all answers on the team round answer sheet.

1. For $x \in \left[0, \frac{\pi}{2}\right]$, determine the <u>largest</u> solution to

   $2\sin x \cdot \cos x \cdot \cos(2x) \cdot \cos(4x) \cdot \cos(8x) \geq \frac{1}{16}$.

2. Given the parabola $y = \frac{x^2}{4p}$ with $p > 0$, let F be the focal point, let M lie on the parabola in the first quadrant with an x-coordinate of a, and let O be the origin. If ΔMOF is isosceles with $OF = OM$, compute the ratio $\frac{a^2}{p^2}$.

3. Compute <u>all</u> ordered pairs (x, y) which satisfy the following system of equations:

$$\begin{cases} x^2 - y^2 = 1 \\ xy - 2y + x = 2 \end{cases}$$

4. Consider two geometric sequences of positive real numbers. The first three terms of the first sequence are the same as the first three terms of the second, <u>but</u> in reverse order. The sum of the first six terms of one sequence is equal to 8 times the sum of the first six terms of the other sequence. If the common ratios are r and m, where $r < m$, compute the ordered pair (r, m).

5. Let $S = \{10, 11, 12, \ldots, 18, 19\}$. If seven numbers are chosen from S at random, compute the probability that the sum of the remaining three numbers is divisible by 10.

6. One term of $\left(x^3 + \frac{1}{x^2}\right)^{12}$ is Px^{11}. Another term of the expansion is Px^n, $n \neq 11$. Find the ordered pair (P, n).

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Answer Sheet

Round 1

1. 6
2. 31
3. ~~$\frac{1}{315}$~~ $\frac{1}{35}$

Round 2

1. 10
2. $\frac{27}{5}$ or $\frac{3}{5}$
3. $4^{100}, 3^{120}$

Round 3

1. 8 : 5 : 2
2. $272 - 96\sqrt{2}$
3. $\frac{a^4}{4a^2 - c^2}$

Round 4

1. $0, 2, -2$
2. $\frac{21}{4}$ or 5.25
3. $165x^{16}$

Round 5

1. $(25, 8, 2, 2)$
2. 72
3. $(\mathbf{3}, \mathbf{2})$

Round 6

1. $(-54\sqrt{3}, 54)$
2. 0.28 or $\frac{7}{25}$
3. (32, 152, 272, 310)

Team

1. $\frac{41\pi}{96}$
2. $4\sqrt{5} - 8$
3. $(2, \pm\sqrt{3}), (\pm\sqrt{2}, -1)$
4. $\left(\frac{1}{2}, 2\right)$
5. $\frac{1}{10}$
6. (792,1)