MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2016 - SOLUTIONS

Round 1 Arithmetic and Number Theory

1. The possible numbers are 4030_9, 4133_9, and 4236_9. Their sum is $\boxed{13410_9}$.

2. Since D is twice C, then $C \le 5$. If $C = 4$ then $D = 8$. Since C is twice B, then $B = 2$, making $A = 1$. This gives 1248 which has ABC = 124 and BCD = 248. C can't be 5 or 3 since that would make $B = 2.5$ or 1.5, nor can $C = 2$ because that makes B = 1 and A would be a fraction. The answers are $\boxed{(1,2,4,8)}$ and $\boxed{(3,7,4,8)}$

3. Since 5 + 3 is not prime, no odd number can work for n. Also, n can't end in 2 since $n+3$ would be divisible by 5. Similarly, n can't end in 8 except for $n = 8$ since otherwise $n-3$ is divisible by 5. We check numbers ending in 0, 4, and 6 and obtain $n = 8$, 10, 14, 16, 20, 26, 34, 40, and 44. Answer: $\boxed{9}$.

Round 2 Algebra 1

1. Since $F = \frac{9}{5}C + 32$, we have $2C = \frac{9}{5}C + 32 \rightarrow C = 160$. Thus, the Fahrenheit reading is $\boxed{320}$.

2. The general case is more interesting. Consider $\frac{1}{a} + \frac{m}{x} = 4$ and $\frac{m}{ax} = 4$. The first gives $\frac{1}{x} = \frac{4a-1}{ma}$ while the second gives $\frac{1}{x} = \frac{4a}{m}$. Then $\frac{4a-1}{ma} = \frac{4a}{m} \rightarrow m's$ cancel with the result that 2016 is irrelevant and $4a - 1 = 4a^2 \rightarrow (2a-1)^2 = 0 \rightarrow \boxed{a = \frac{1}{2}}$.

3. Let $Y = \frac{x^2+3}{x}$. Squaring both sides, $\sqrt{Y} - \sqrt{\frac{1}{Y}} = \frac{3}{2} \Rightarrow Y - 2 + \frac{1}{Y} = \frac{9}{4}$

$$\Rightarrow Y + \frac{1}{Y} = \frac{17}{4} \Rightarrow 4Y^2 - 17Y + 4 = (4Y-1)(Y-4) = 0 \Rightarrow Y = \frac{1}{4}, 4.$$

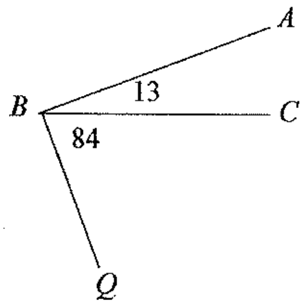
$\frac{x^2+3}{x} = \frac{1}{4} \Rightarrow 4x^2 - x + 12 = 0$ which has no real roots.

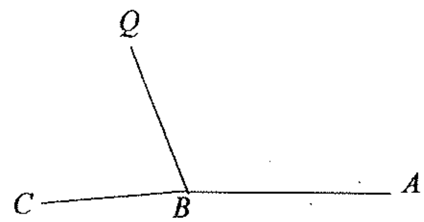
$\frac{x^2+3}{x} = 4 \Rightarrow x^2 - 4x + 3 = (x-1)(x-3) = 0 \Rightarrow x = \underline{\mathbf{1,3}}$

Both values check in the original equation.

Round 3 – Geometry

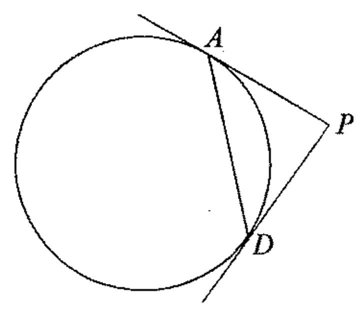
1. As shown in the left-hand diagram, $\angle ABC = m\angle ABQ - m\angle CBA = 97 - 84 = 13$. As shown in the right-hand diagram, $m\angle ABC + m\angle ABQ + m\angle CBQ = 360$, so $m\angle ABC = 360 - 97 - 84 = 179$. Answer: $\boxed{13 \text{ and } 179}$.





2. The area of $\Delta \boldsymbol{MCN}$ is $\frac{1}{2}\mathbf{20 \cdot 15}$=150. The volume of pyramid $\boldsymbol{MCNB}$ is $\frac{1}{3}\mathbf{150 \cdot 30}$= 1500. The required volume is $\mathbf{30^3 - 1500 =}\boxed{\mathbf{25500}}$.

3. Let $m\angle A = x$, then $m\overset{\frown}{AD} = 8x$, $m\overset{\frown}{BC} = 2x$, $m\angle ACD = 4x$, making $m\angle P = \dfrac{8x - 2x}{2} = 3x$. If $\overline{PAB}$ and $\overline{PCD}$ were rotated, keeping P fixed, until the lines became tangents, then A and B would coincide, and C and D would coincide as shown in the diagram. Since $m\angle P + m\overset{\frown}{AD} = 180$, for minor arc $\overset{\frown}{AD}$, we have $2x + 3x = 180 \rightarrow x = 36$. This would make $m\angle P = 108$, but the lines must be secants, not tangents, so the largest possible integer measure of $\angle P$ is $\boxed{107}$.



Round 4 – Algebra 2

1. Let $Y = \log_6 x$.

$$(\log_6 x)^2 + 3\log_6(6x) - \frac{1}{2}\log_{\sqrt{6}} 6 = 0 \Leftrightarrow Y^2 + 3(1+Y) - 1 = 0$$

$$\Leftrightarrow Y^2 + 3Y + 2 = (Y+1)(Y+2) = 0 \Rightarrow Y = -1, -2 \Rightarrow x = \underline{\mathbf{\frac{1}{6}, \frac{1}{36}}}$$

2. If f is its own inverse then $f\left(f\left(x\right)\right)=x$ so $f\left(\frac{ax}{x+2}\right)=x\rightarrow\frac{a\left(\frac{ax}{x+2}\right)}{\frac{ax}{x+2}+2}=x\rightarrow$

$\frac{a^2x}{x+2}=\frac{ax^2}{x+2}+2x\rightarrow(a+2)x^2+(4-a^2)x=0$. This is true for all x in the domain of f if $\boxed{a=-2}$.

3. Assume the equation of the given line is $y=mx+b$.

Since the x-intercept is $-\frac{b}{m}$, $-\frac{b}{m},m,b$ is an arithmetic sequence with a common difference d.

Thus, $d=\frac{15}{2}=b-m=m+\frac{b}{m}\Rightarrow\begin{cases}m=b-\frac{15}{2}\\2m^2+b=mb\end{cases}$

Substituting, $2\left(b-\frac{15}{2}\right)^2+b=\left(b-\frac{15}{2}\right)b$

Multiplying through by 2 and expanding, $\Rightarrow 4b^2-60b+225+2b=2b^2-15b$

$\Leftrightarrow 2b^2-43b+225=0\Leftrightarrow(2b-25)(b-9)=0\Rightarrow b=\underline{\frac{25}{2}}, 9$

Round 5 – Analytic Geometry

1. From $-4x+7=2x^2-4x-1$, $x=-2$ or $x=2$. The points of intersection are $(-2,15)$ and $(2,-1)$. The slope is -4. The required slope is $\frac{1}{4}$.

2. $\pi ab=100\pi$, $\varepsilon=0.6=\sqrt{\frac{a^2-b^2}{a^2}}\rightarrow a^2-b^2=0.36a^2\rightarrow b^2=0.64a^2\rightarrow b=0.8a$.

Substituting into the first equation gives $0.8a^2=100\rightarrow a=5\sqrt{5}\rightarrow b=4\sqrt{5}$.

Focal length is $\frac{2b^2}{a}=\frac{2\cdot 80}{5\sqrt{5}}=\boxed{\frac{32\sqrt{5}}{5}}$.

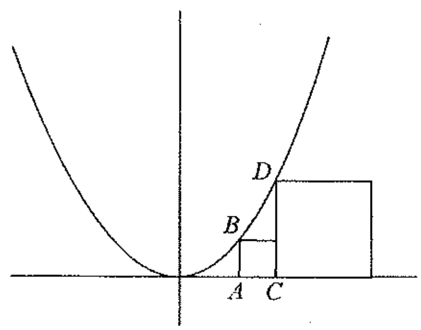
Alternate Solution:

Let $a=5k$, $c=3k$, then $b=4k$. Then $20k^2=100$ so $k=\sqrt{5}$. The equation of the ellipse is $\frac{x^2}{125}+\frac{y^2}{80}=1$. Substituting $x=3\sqrt{5}$, we get $\frac{y^2}{80}=1-\frac{45}{125}=1-\frac{9}{25}=\frac{16}{25}$. Then $y=\pm\frac{4}{5}\cdot 4\sqrt{5}=\pm\frac{16}{5}\sqrt{5}$. so the focal chord is $\boxed{32\frac{\sqrt{5}}{5}}$.

3. Let $A=(x,0)$, then $B=\left(x,x^2\right)$, making $C=\left(x+x^2,0\right)$ and $D=\left(x+x^2,\left(x+x^2\right)^2\right)$.

Then the area of the smaller square is $\left(x^2\right)^2=x^4$. The area of the larger square is

$\left(\left(x+x^2\right)^2\right)^2=\left(x+x^2\right)^4$. Then $\dfrac{\left(x+x^2\right)^4}{x^4}=81\rightarrow\dfrac{\left(x+x^2\right)}{x}=3\rightarrow x^2-2x=0$. Thus, $x=2$ so the side of the smaller square is $\boxed{4}$.



Round 6 – Trig and Complex Numbers

1. If n is a multiple of 7, then the expression either equals $\cos\pi+i\sin\pi=-1$ or $\cos 2\pi+i\sin 2\pi=1$. Since $\dfrac{1000}{7}=142.857\square$, there are $\boxed{142}$ values of n that give real numbers.

2. Let $\sin^{-1}x=\theta$, then $\tan\theta=2$ and $\sin\theta=x$. First, for the tangent to be positive, x must be a first quadrant angle. Second, $\cos\theta=\sqrt{1-x^2}$, giving $\dfrac{x}{\sqrt{1-x^2}}=2\rightarrow x^2=4-4x^2$. Hence, $x^2=\dfrac{4}{5}\rightarrow\boxed{x=\dfrac{2\sqrt{5}}{5}}$.

3. By the Law of Cosines we have $\left(\dfrac{1}{2}\right)^2=\sin^2 A+\cos^2 A-2\sin A\cos A\cos A=$ $\dfrac{1}{4}=1-2\cos^2 A\sin A\rightarrow 2\cos^2\sin A=\dfrac{3}{4}\rightarrow\left(1-\sin^2 A\right)\sin A=\dfrac{3}{8}\rightarrow\sin^3 A-\sin A+\dfrac{3}{8}=0$. We're told that $\dfrac{1}{2}$ is a solution. The reduced polynomial is $4\sin^2 A+2\sin A-3=0$. The only positive solution is $\dfrac{-1+\sqrt{13}}{4}$. Then $\boxed{(a,b,c)=(-1,13,4)}$.

Team Round

1. $\dfrac{4}{3x}=\dfrac{y}{2}-8 \Rightarrow y=2\left(\dfrac{4}{3x}+8\right)=\boxed{\dfrac{8(6x+1)}{3x}}$ Substituting in the first equation,

$\dfrac{9x}{2}+\dfrac{7}{4}\cdot\dfrac{3x}{8(6x+1)}+\dfrac{9}{16}=0$ Multiplying by $\dfrac{2}{3}$, $3x+\dfrac{7x}{16(6x+1)}+\dfrac{3}{8}=0$

Multiplying by the LCD $\left(16(6x+1)\right)$, $48x(6x+1)+7x+6(6x+1)=0$

$\Leftrightarrow 288x^2+91x+6=0$

With an odd middle term, we try factoring 288 as an odd times an even $\left(~~288\cdot 1~~, ~~96\cdot 3~~, 32\cdot 9\right)$

$(32x+3)(9x+2)=0 \Rightarrow x=-\dfrac{3}{32},-\dfrac{2}{9}$ Substituting in the boxed expression,

$$\frac{8\left(6\cdot\frac{-3}{32}+1\right)}{3\cdot\frac{-3}{32}}=\frac{8(-18+32)}{-9}=-\frac{112}{9} \text{ and } \frac{8\left(6\cdot\frac{-2}{9}+1\right)}{3\cdot\frac{-2}{9}}=\frac{8(-12+9)}{-6}=\frac{8\cdot 3}{6}=4.$$

Thus, we have $(x,y)=\underline{\left(-\mathbf{\dfrac{3}{32}},-\mathbf{\dfrac{112}{9}}\right),\left(-\mathbf{\dfrac{2}{9}},\mathbf{4}\right)}$.

Alternate Solution:

The second equation can be written as $\dfrac{4}{3x}=\dfrac{y-16}{2}$. Multiplying the first equation by 1/6, we get $\dfrac{3x}{4}+\dfrac{7}{24y}+\dfrac{3}{32}=0$. Combining gives $\dfrac{2}{y-16}=-\dfrac{7}{24y}-\dfrac{3}{32}\rightarrow 9y^2+76y-448=0$. Reducing the roots by a factor of 4, we get $9y^2+\dfrac{76y}{4}-\dfrac{448}{16}=0$ $9y^2+19y-28=0$ where $y=1$ is a root. So the factors are $(y-1)(9y+28)=0$, then the roots of the original equation are $y=4$ and $y=-\dfrac{112}{9}$. Substituting into $\dfrac{1}{x}=\dfrac{3(y-16)}{8}$ gives $x=-\dfrac{2}{9}$ and $x=-\dfrac{3}{32}$.

2. Squaring PQ, $(x-3)^2+(y-7)^2=65$

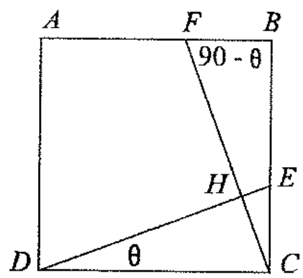
The only possible sums of perfect squares that produce 65 are 1^2+8^2 and 4^2+7^2 .
$x-3=1,4,7,8 \Rightarrow x>0$ All are rejected.

$$x-3=\begin{cases}-1\\-4\\-7\\-8\end{cases}\Rightarrow x=\begin{cases}\cancel{2}\\-1\\-4\\-5\end{cases}\Rightarrow$$

$$(y-7)^2=65-(x-3)^2=65-\begin{cases}16\\49\\64\end{cases}=\begin{cases}49\\16\\1\end{cases}\Rightarrow y=7+\begin{cases}\pm7\\\pm4\\\pm1\end{cases}\Rightarrow y=\begin{cases}14\\3,\ 11\\6,\ 8\end{cases}$$

Thus, we have 5 ordered pairs, namely, $\underline{\mathbf{(-1,14),(-4,3),(-4,11),(-5,6),(-5,8)}}$

3. The area of $ABCD$ is 3600 and the sum of the areas of ΔBCF and ΔCDE is $2\cdot\frac{1}{2}\cdot60\cdot20=1200$. This counts the area of EHC twice so it must be subtracted. Let $m\angle EDC=\theta$, then $m\angle DEC=90-\theta$, and since $\Delta BCF\cong\Delta CDE$, then $m\angle BCF=\theta$, making $\Delta CHE \sim \Delta DCE$.
$DE=\sqrt{60^2+20^2}=20\sqrt{10}$, so the ratio of EC to DC is $\frac{20}{20\sqrt{10}}=\frac{1}{\sqrt{10}}$ and since the ratio of areas is the square of the ratio of corresponding sides, the area of ECH equals $\frac{1}{10}\cdot\frac{1}{2}\cdot20\cdot60=60$. The area of $AFHD=3600-(1200-60)=\boxed{2460}$.



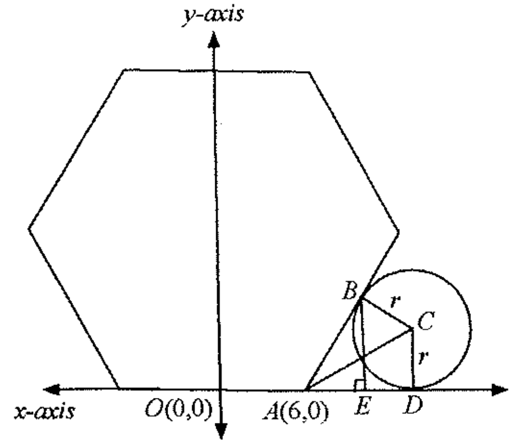
Alternate Solution:

Put the square on a coordinate system with $D(0,0), C(60,0), B(60,60), A(0,60)$. Then $F(40,60)$ DE is $x-3y=0$, and CF is $3x+y=180$. Solving we get $H=(54,\ 18)$. Then using the determinant method, we get

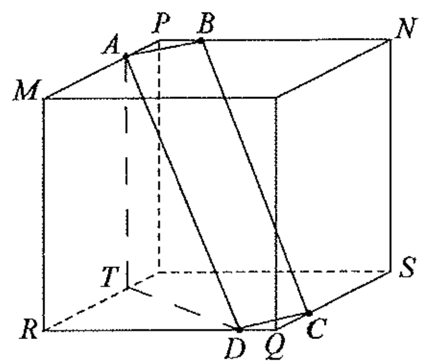
$$.5\left\|\begin{matrix}0&0\\54&18\\40&60\\0&60\\0&0\end{matrix}\right\|=$$

0.5(3240 + 2400 −720) = 2460.

4. Since $m\angle BAC = 60$ and $\overline{AC}$ bisects the angle, ABC and ADC are 30-60-90 right triangles, making $AD = AB = r\sqrt{3}$. Drop a perpendicular from the point of tangency B. Then $AE = \frac{r\sqrt{3}}{2}$ making $BE = \frac{r\sqrt{3}}{2} \cdot \sqrt{3} = \frac{3r}{2}$. Since $r = 4$ the coordinates of B are $(AE, BE) = \boxed{(2\sqrt{3}, 6)}$.



5. Let the edge of the cube equal $3x$. Then $AP = BP = x$, making $AB = x\sqrt{2}$. Let T lie directly below A. Then $TR = 2x = RD$, making $TD = 2x\sqrt{2}$. Since $AT = 3x$, then $AD = \sqrt{9x^2 + 8x^2} = x\sqrt{17}$. The area of $ABCD$ is, therefore $x\sqrt{2} \cdot x\sqrt{17} = x^2\sqrt{34}$. Setting $x^2\sqrt{34} = \sqrt{17}$ gives $x^2 = \frac{1}{\sqrt{2}}$. If the edge of the cube is $3x$, then the surface area is $6(3x)^2 = 54x^2$. The surface area is, therefore, $54 \cdot \frac{1}{\sqrt{2}} = \boxed{27\sqrt{2}}$



6. Using the reciprocal relationship and the power rule, we have $\frac{1}{\log_y x} + 2\log_y x - 3 = 0 \rightarrow$

$2(\log_y x)^2 - 3\log_y x + 1 = 0 \rightarrow (2\log_y x - 1)(\log_y x - 1) = 0$. Thus, $x = y^{1/2}$ or $y = x \rightarrow y = x^2$ or $y = x$. The first gives the two ordered pairs (2,4) and (3, 9). Since a log base can't be 1, the second gives 8 ordered pairs from (2, 2) to (9, 9). The answer is $\boxed{10}$.