MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2018 - SOLUTIONS

Round 1: Arithmetic and Number Theory

1. $AB = \frac{2^3 \cdot 3^3}{5^2} \cdot \frac{5^3}{2^2 \cdot 3} = 2 \cdot 3^2 \cdot 5 = 90$. Then $90 = 1122_4$

2. $\left(.\overline{72}\right)^2 - \left(.\overline{27}\right)^2 = \left(.\overline{72} + .\overline{27}\right)\left(.\overline{72} - .\overline{27}\right) = \left(.\overline{9}\right)\left(\frac{8}{11} - \frac{3}{11}\right) = \frac{5}{11}$. The sum is $\boxed{16}$

3. Since n cannot have a factor of 2, 3, 5, or 7, n can equal 13, 17, 19, 23, or 29.

   Since the product equals 840, here are the possible values of P: 853, 857, 859, 863, 869.

   Check these for divisibility by a prime greater than 7. Note that 869 satisfies the divisibility rule for 11, namely that $8 + 9 - 6 = 11$. Thus, we find that $869 = 11 \cdot 79$, so we reject it. We check the others for divisibility by 13, 17, 19, 23, and 29 and none are divisible by those numbers, giving $\boxed{4}$ values for n.

Round 2 Algebra 1

1. From the first equation, $x = 2$. From the second equation, y = 6 or -2. Then $2^{-3} \cdot y^2 = \frac{36}{8} = \frac{9}{2}$, or $2^{-3} \cdot y^2 = \frac{4}{8} = \frac{1}{2}$. The positive difference is 4.

2. If $x > 0$, $|x| = x$ $\frac{x}{x-1} = \frac{x+1}{x} \rightarrow x^2 = x^2 - 1 \rightarrow 0 = -1$, so no solution. If $x < 0$ then $|x| = -x$, giving $\frac{-x}{-x-1} = \frac{x+1}{x} \rightarrow x^2 = x^2 + 2x + 1$ making $\boxed{x = -\frac{1}{2}}$

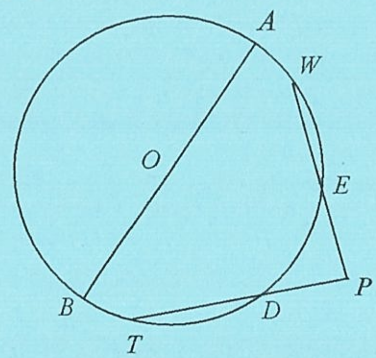
3. $k(x+3) = \frac{1}{1 + \frac{1}{x+3}} = \frac{x+3}{x+4} \rightarrow k = \frac{1}{x+4} \rightarrow x = \frac{1}{k} - 4$. Then $16 \le \frac{1}{k} - 4 \le 26 \rightarrow \frac{1}{30} \le k \le \frac{1}{20}$.

   Then $300\left(\frac{1}{20} - \frac{1}{30}\right) = \boxed{5}$. Note: x can't be -3 since it makes a denominator equal to 0.

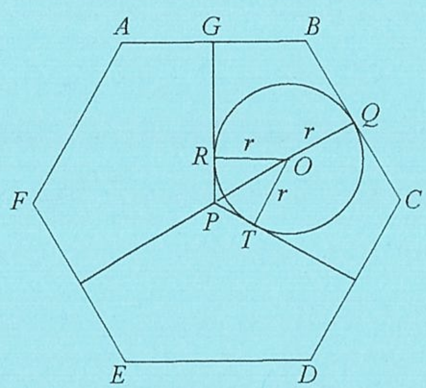
Round 3 – Geometry

1. The altitude of the triangle is 10 and a diagonal of the square is $10\sqrt{2}$. The areas of the circles are 25π and 50π. The positive difference is 25π.

2. Since $m\angle P = \dfrac{m\widehat{TBAW} - 30}{2}$ and $m\widehat{TBAW} > 180$, then $\boxed{m\angle P > 75^\circ}$.



3. Since $m\angle OPT = 60$, $PT = \dfrac{r}{\sqrt{3}}$ and $OP = \dfrac{2r}{\sqrt{3}}$. Since $\overline{PQ}$ is the altitude of equilateral triangle BPC and $BC = 6$, then $PQ = 3\sqrt{3}$. Setting $r + \dfrac{2r}{\sqrt{3}} = 3\sqrt{3}$ gives $r\sqrt{3} + 2r = 9 \rightarrow \boxed{r = 18 - 9\sqrt{3}}$.



Round 4 – Algebra 2

1. $3 \cdot \frac{1}{15} + 3 \cdot \frac{1}{9} = \frac{8}{15}$ of the job done in 3 hours, which means $\frac{7}{15}$ of the job remains to be done. $\left(\frac{1}{15} + \frac{1}{9} + \frac{1}{6}\right) x = \frac{7}{15} \rightarrow x = \frac{42}{31}$

2. Factor the second equation as $y\left(3x^2 - y^2\right) = 0$. These are the solutions: $y = 0$ or $y^2 = 3x^2$. Substituting $y = 0$ into the first equation gives $x^3 = 125 \rightarrow x = 5$. Substituting $y^2 = 3x^2$ into the first equation gives $x^3 - 9x^3 = 125 \rightarrow 8x^3 = -125 \rightarrow x^3 = -\frac{125}{8}$. Thus, $x = -\frac{5}{2}$. Then $y^2 = \frac{75}{4} \rightarrow y = \pm\frac{5\sqrt{3}}{2}$. Answers: $\boxed{(5,0), \left(-\frac{5}{2}, \frac{5\sqrt{3}}{2}\right), \left(-\frac{5}{2}, -\frac{5\sqrt{3}}{2}\right)}$

3. Since f is odd and of the 5th degree, we have $f(x) = ax^5 + bx^3 + cx$. Also, $f(6) = 7776a + 216b + 6c = (7770a + 6a) + (210b + 6b) + 6c$. Dividing by 10 will yield remainders of 0 with 7770a and 210a, so the remainder is determined by $6a + 6b + 6c$. But this equals $6(a + b + c) = 6f(1) = 6 \cdot 7 = 42$. The remainder when this is divided by 10 is $\boxed{2}$.

Round 5 – Analytic Geometry

1. The equation can be rewritten as $4(y^2 - 12y + 36) - 9(x^2 + 4x + 4) = 36$. This can further be rewritten as $\frac{(y-6)^2}{9} - \frac{(x+2)^2}{4} = 1$. This represents a hyperbola whose center is $(-2, 6)$. To find the x-intercepts, let y=0, which gives $4 - \frac{(x+2)^2}{4} = 1 \rightarrow (x + 2)^2 = 12 \rightarrow x = -2 \pm 2\sqrt{3}$. Therefore, the distance between the x-intercepts is $4\sqrt{3}$. The required area is $\frac{1}{2} \cdot 6 \cdot 4\sqrt{3}$ or $12\sqrt{3}$.

2. The equation of the line through (1,1) perpendicular to $y = 2x$ is $y - 1 = -\frac{1}{2}(x - 1) \rightarrow y = -\frac{x}{2} + \frac{3}{2}$. Its intersection with $y = 2x$ is the point $\left(\frac{3}{5}, \frac{6}{5}\right)$. This means that the vector $\left\langle \frac{3}{5} - 1, \frac{6}{5} - 1 \right\rangle = \left\langle -\frac{2}{5}, \frac{1}{5} \right\rangle$ takes (1,1) to $y = 2x$, so it will take $\left(\frac{3}{5}, \frac{6}{5}\right)$ to the reflection point. Answer: $\boxed{\left(\frac{1}{5}, \frac{7}{5}\right)}$.

3. Let the x-intercept be a. The slope of the line is $\frac{1-0}{2-a}$, the equation of the line is

$y-0=\frac{1}{2-a}(x-a)$ and if $x=0$, the y-intercept is $\frac{a}{a-2}$. Then

$\frac{1}{2}\cdot a\cdot\frac{a}{a-2}=6\rightarrow a^2-12a+24=0\rightarrow a=6+2\sqrt{3}$, making

$y=\frac{6+2\sqrt{3}}{4+2\sqrt{3}}=\frac{3+\sqrt{3}}{2+\sqrt{3}}\cdot\frac{2-\sqrt{3}}{2-\sqrt{3}}=3-\sqrt{3}$. The sum of the intercepts is $\boxed{9+\sqrt{3}}$.

Round 6 – Trig and Complex Numbers

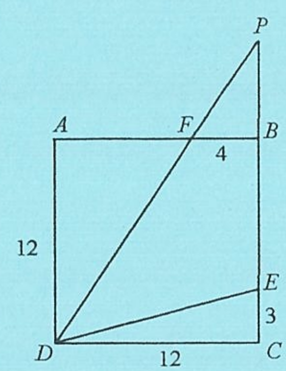
1. $1-2\sin^2\theta=\frac{7}{25}\longrightarrow -2\sin^2\theta=-\frac{18}{25}\rightarrow \sin^2\theta=\frac{9}{25}\rightarrow \sin\theta=\frac{3}{5}$. Now $\cos\theta=-\frac{4}{5}$. This gives us $\sin(\theta+45°)=\frac{3}{5}\cdot\frac{\sqrt{2}}{2}+\left(-\frac{4}{5}\right)\cdot\frac{\sqrt{2}}{2}=\frac{-\sqrt{2}}{10}$.

2. Let $C=(3,0)$. Then $BC=4$ and ΔBAC is a 3-4-5 right triangle. Let $m\angle BAC=\theta$. Call the point that B rotates to $D\ (x,y)$. Then $\sin\angle DAC=\sin(\theta+30°)=\frac{y}{5}$. Thus, $\sin\theta\cos 30+\cos\theta\sin 30=\frac{y}{5}$ giving

$\frac{4}{5}\cdot\frac{\sqrt{3}}{2}+\frac{3}{5}\cdot\frac{1}{2}=\frac{y}{5}\rightarrow \boxed{y=\frac{4\sqrt{3}+3}{2}}$.

3. Extend $\overline{DF}$ and $\overline{CB}$ until they meet at P. Let $PB=x$. Since $\Delta PFB \sim \Delta PDC$, $\frac{PB}{FB}=\frac{PC}{DC}\rightarrow\frac{x}{4}=\frac{x+12}{12}\rightarrow x=6$. Let $m\angle EDC=\alpha$ and $m\angle PDC=\beta$. Then $\tan\angle FDE=\tan(\beta-\alpha)=\frac{\tan\beta-\tan\alpha}{1+\tan\beta\cdot\tan\alpha}$. Since $\tan\alpha=\frac{3}{12}=\frac{1}{4}$ and $\tan\beta=\frac{18}{12}=\frac{3}{2}$, then

$\tan\angle FDE=\frac{\frac{3}{2}-\frac{1}{4}}{1+\frac{3}{2}\cdot\frac{1}{4}}=\boxed{\frac{10}{11}}$.

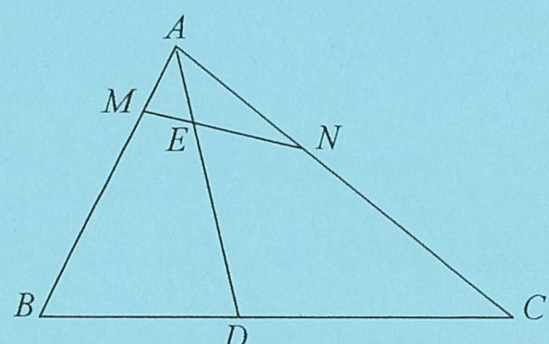


Team Round

1. $\cos\left(\cos^{-1}x+\cos^{-1}x\right)=\cos\left(\cos^{-1}\frac{x}{2}\right)\to\cos\left(\cos^{-1}x\right)\cos\left(\cos^{-1}x\right)-\sin\left(\cos^{-1}x\right)\sin\left(\cos^{-1}x\right)$ $=\frac{x}{2}$. Thus, $x^2-\left(\sqrt{1-x^2}\right)^2=\frac{x}{2}\to 2x^2-\frac{x}{2}-1=0$. Then $x=\frac{1\pm\sqrt{33}}{8}$. However, if we think about the graphs we realize that $y=2\cos^{-1}x$ is a decreasing function from 2π to 0 on $[-1,1]$ and $y=\cos^{-1}(.5x)$ is a decreasing function from π to 0 on $[-2,2]$ and so they can intersect only at one point and that has to be in the first quadrant. Answer: $\boxed{\frac{1+\sqrt{33}}{8}}$.

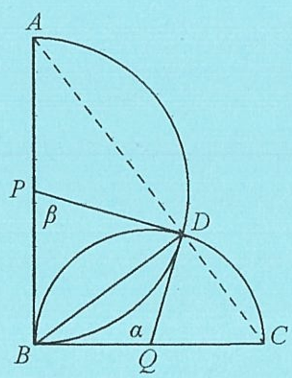
2. Note that if $\frac{a}{b}=\frac{c}{d}$, setting $c=am$ and $d=bm$ gives $\frac{a+c}{b+d}=\frac{a+am}{b+bm}=\frac{a(1+m)}{b(1+m)}=\frac{a}{b}$. Thinking of 12 as $9m$ and $(3-y)$ as $(x+y)m$ we have $\frac{9}{x+y}=\frac{9+12}{(x+y)+(3-y)}=\frac{21}{x+3}$. Thus, $n=\boxed{21}$.

3. Let $m\angle BAC=\theta$. Then the area of $\Delta AMN=\frac{1}{2}\cdot 1\cdot 2\sin\theta=\sin\theta$ and the area of $\Delta ABC=$ $\frac{1}{2}\cdot 4\cdot 6\sin\theta=12\sin\theta=K$. Then the area of $\Delta AMN=\frac{K}{12}$. Since $\overline{AE}$ bisects $\angle BAC$, the ratio of ME to EN equals the ratio of AM to AN = 1:2 so the area of ΔAME is $\frac{1}{3}\cdot\frac{K}{12}=\frac{K}{36}$ and the area of ΔAEN is $\frac{2}{36}K$. Since $\overline{AE}$ bisects $\angle BAC$, the ratio of BD to DC is 4:6 = 2:3, so the area of ΔABD equals $\frac{2}{5}K$. Thus, the area of $MEDB$ equals $\frac{2}{5}K-\frac{K}{36}=\frac{67K}{180}$. The ratio of the area of ΔAEN to the area of $MEBD=\frac{2/36}{67/180}=\boxed{\frac{10}{67}}$.



4. Let D be the intersection of the two arcs, let P and Q be the centers of the two semicircles, and draw $\overline{PD}$, $\overline{QD}$, $\overline{BD}$, and $\overline{AC}$. The sum of the areas of the two semicircles is $\frac{1}{2}\pi\left(\left(\sqrt{3}\right)^2+1^2\right)=2\pi$. We must subtract the common region since it is counted twice. Note that $m\angle BCD=60$ and since $QC=QD$, then QDC is an equilateral triangle making $\alpha=120$. Thus, the area of segment $\overline{BD}$ of the small circle is $\frac{1}{3}\pi\cdot 1^2-\frac{1}{2}\cdot 1\cdot 1\cdot\sin 120 = =\frac{\pi}{3}-\frac{\sqrt{3}}{4}$

Similarly, we show that $\beta=60$ so the area of segment $\overline{BD}$ of the large circle is $\frac{1}{6}\pi\left(\sqrt{3}\right)^2-\frac{1}{2}\cdot\sqrt{3}\cdot\sqrt{3}\sin 60=\frac{\pi}{2}-\frac{3\sqrt{3}}{4}$. Thus, the area of the shaded region is $2\pi-\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}+\frac{\pi}{2}-\frac{3\sqrt{3}}{4}\right)=\boxed{\frac{7\pi}{6}+\sqrt{3}}$.



Alternate solution: If we place the diagram on a coordinate axis system with B as the origin then the equations of the two circles are $(x-1)^2+y^2=1$ and $x^2+\left(y-\sqrt{3}\right)^2=3$. Solving we obtain $y=\frac{x}{\sqrt{3}}$ which gives $x^2-2x+\frac{x^2}{3}=0\rightarrow x=\frac{3}{2}$ making $y=\frac{\sqrt{3}}{2}$. Then $DC=\sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\sqrt{3}$. Using the Law of Cosines we find that $\alpha=120$ and $\beta=60$ and the solution proceeds as above.

5. $\frac{a_1+a_n}{2}=k\rightarrow\frac{a_1+[a_1+(n-1)1]}{2}=k\rightarrow 2a_1+n-1=2k$.

$\frac{a_2+a_n}{2}=2k\rightarrow\frac{(a_1+1)+(a_1+n-1)}{2}=2k\rightarrow 2a_1+n=4k$. Then $4a_1+2n-2=2a_1+n\rightarrow$ $2a_1+n=2$. Since $n\geq 1$, then $a_1<1$. If $a_1=0$, then $n=2$, and the consecutive integers are 0 and 1. The average is 1/2 and if 0 is removed the average is 1. If, for example, $a_1=-1$, then $n=4$, the numbers are –1, 0, 1, 2. The average is again 1/2 and if –1 is removed, the average is 1. If $a_1=-2$, then $n=6$, the numbers are –2, –1, 0, 1, 2, 3. The average is 1/2 and if –2 is removed, the average is 1. Clearly, the largest possible value of the smallest number a_1 in a set is $\boxed{0}$.

6. We have 4 cases: $\left\lfloor\frac{1}{x}\right\rfloor=1$ and $\left\lfloor\frac{1}{y}\right\rfloor=8$, $\left\lfloor\frac{1}{x}\right\rfloor=2$ and $\left\lfloor\frac{1}{y}\right\rfloor=4$, $\left\lfloor\frac{1}{x}\right\rfloor=4$ and $\left\lfloor\frac{1}{y}\right\rfloor=2$, and $\left\lfloor\frac{1}{x}\right\rfloor=8$ and $\left\lfloor\frac{1}{y}\right\rfloor=1$. Analyzing one of them ought to provide the clues to the others. In the first case, from $\left\lfloor\frac{1}{x}\right\rfloor=1$ and $\left\lfloor\frac{1}{y}\right\rfloor=8$, we obtain $1\leq\frac{1}{x}<2$, giving $\frac{1}{2}<x\leq 1$ and $8\leq\frac{1}{y}<9$, giving $\frac{1}{9}<y\leq\frac{1}{8}$. The solution set to these conditions would be a rectangle in the coordinate plane with width $1-\frac{1}{2}=\frac{1}{2}$ and height $\frac{1}{8}-\frac{1}{9}=\frac{1}{72}$. The bottommost and leftmost boundaries of the rectangle would not be included in the solution set, but not counting those segments will not affect the area of the region. The area is $\frac{1}{2}\cdot\frac{1}{72}=\frac{1}{144}$. The area determined by $\left\lfloor\frac{1}{x}\right\rfloor=8$ and $\left\lfloor\frac{1}{y}\right\rfloor=1$ will also be $\frac{1}{144}$. In similar fashion, the region determined by $\left\lfloor\frac{1}{x}\right\rfloor=2$ and $\left\lfloor\frac{1}{y}\right\rfloor=4$ will be the rectangle where $\frac{1}{3}<x\leq\frac{1}{2}$ and $\frac{1}{5}<y\leq\frac{1}{4}$. This rectangle will have dimensions $\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$ and $\frac{1}{4}-\frac{1}{5}=\frac{1}{20}$ and its area will be $\frac{1}{120}$, the same as the area of the region determined by $\left\lfloor\frac{1}{x}\right\rfloor=4$ and $\left\lfloor\frac{1}{y}\right\rfloor=2$. The sum of the areas is $2\cdot\frac{1}{144}+2\cdot\frac{1}{120}=\frac{1}{72}+\frac{1}{60}=\frac{1}{12}\left(\frac{1}{6}+\frac{1}{5}\right)=\frac{11}{360}$. Then $11+360=\boxed{371}$.