MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2014 - SOLUTIONS

Round 1 Arithmetic and Number Theory

1. $$1+\cfrac{1}{.\overline{1}+\cfrac{.\overline{8}}{.\overline{2}+\cfrac{.\overline{7}}{.\overline{3}+\cfrac{.\overline{6}}{.\overline{4}+.\overline{5}}}}} = 1+\cfrac{1}{.\overline{1}+\cfrac{.\overline{8}}{.\overline{2}+\cfrac{.\overline{7}}{.\overline{3}+.\overline{6}}}} = 1+\cfrac{1}{.\overline{1}+\cfrac{.\overline{8}}{.\overline{2}+.\overline{7}}} = 1+\frac{1}{.\overline{1}+.\overline{8}} = \boxed{2}.$$

2.

Dimes	Nickels	Pennies	
2	0	3	There are $\boxed{9}$ ways to make \$0.23 in change
1	2	3	
0	4	3	
1	1	8	
1	0	13	
0	3	8	
0	2	13	
0	1	18	
0	0	23	

3. 1111 to 2110 has 1000_6 numbers for a total of 216. 2111 to 2510 has 400_6 for a total of 144. 2511 to 2562 has a total of $52_6 = 32$ numbers. The total is 216 + 144 + 32 = $\boxed{392}$.

Round 2 Algebra 1

1. Express $\dfrac{x^2+5x+2}{x^2-9}$ as $\dfrac{x^2+5x+2}{(x-3)(x+3)} \approx \dfrac{9+15+2}{(.000001)6} = \dfrac{26}{(.000001)6} = 1{,}000{,}000 \cdot 4\frac{1}{3} =$ $4{,}333{,}333$ so the answer would be $\boxed{4{,}300{,}000}$.

2. Rewrite the second equation as $x-y=4$ and add to the first equation, obtaining $x^2+x-12=0$. From $(x+4)(x-3)=0$ we obtain $x=-4$ or 3, giving $(x,y)=(-4,-8),(3,-1)$. The largest possible product is $\boxed{32}$.

3. $\frac{k}{x-y} = \frac{1}{x} - \frac{1}{y} = \frac{y-x}{xy} \rightarrow kxy = -(x-y)^2 \rightarrow x^2 + (k-2)xy + y^2 = 0$. Treat this equation as a quadratic in x. If the discriminant is positive then there are solutions for x and y. Then $x = \frac{(2-k)y \pm \sqrt{(k-2)^2 y^2 - 4y^2}}{2} = \frac{(2-k)y \pm |y|\sqrt{(k-2)^2 - 4}}{2}$. There will be real values for x and therefore for y if $(k-2)^2 - 4 \geq 0$. From $k^2 - 4k \geq 0$ we obtain $\boxed{k \leq 0 \text{ or } k \geq 4}$.

Round 3 – Geometry

1. $6^3 - \frac{4}{3}\pi \cdot 3^3 = 216 - 36\pi$

2. Since the sides must be positive we have $x^2 + 3x > 0 \rightarrow x < -3$ or $x > 0$. Similarly, $x^2 + x > 0 \rightarrow x < -1$ or $x > 0$. From the triangle inequality we obtain

(1) $\left(x^2 + 3x\right) + \left(x^2 + x\right) > 16 \rightarrow x^2 + 2x - 8 > 0 \rightarrow x < -4$ or $x > 2$.

(2) $\left(x^2 + x\right) + 16 > x^2 + 3x \rightarrow x < 8$.

(3) $\left(x^2 + 3x\right) + 16 > x^2 + x \rightarrow x > -8$.

The intersection of the four solution sets is $\boxed{-8 < x < -4 \text{ or } 2 < x < 8}$. This could be written as $\boxed{(-8,-4) \cup (2,8)}$.

3. Let k be the scaling factor giving a triangle with sides $3k$, $9k$, and $10k$. The perimeter is $22k$. By Heron's formula the area is $\sqrt{(11k)(11k-3k)(11k-9k)(11k-10k)} = 4k^2\sqrt{11}$. From $4k^2\sqrt{11} = 22k$ we obtain $\boxed{k = \frac{\sqrt{11}}{2}}$.

Round 4 – Algebra 2

1. $f(-2) = \frac{-3}{-1} = 3$, $(3) = \frac{2}{4} = \frac{1}{2}$. $f\left(\frac{1}{2}\right) = \frac{\frac{1}{2}-1}{\frac{1}{2}+1} = -\frac{1}{3}$, $f\left(-\frac{1}{3}\right) = \frac{-\frac{1}{3}-1}{-\frac{1}{3}+1} = \frac{-\frac{4}{3}}{\frac{2}{3}} = -2$
$\therefore f^4(-2) = -2$. For n divisible by 4, $f^n(-2) = -2$. $\therefore f^{2012}(-2) = -2$. $f^{2014}(-2) = f^2(-2) = f(f(-2)) = \boxed{\frac{1}{2}}$.

2. Solution 1: The number of possible pairs of factors of 60 and –60 will give the result. Hence 60,1; 30,2; 20,3; 4,15; 5,12; 6,10. Hence $\boxed{12}$ possible values of k.

Solution 2: Try all the factor combinations for $ab = 4$ and $cd = 15$ in $(ax - c)(bx - d)$.

Solution 3:
Suppose for integers a, b, c, and d, $4x^2 - kx + 15 = (ax-b)(cx-d) = acx^2 - (ad+bc)x + bd$. We must consider all possible quadruples (a,b,c,d) for which $(ac)(bd) = 4(15) = 60$. There are exactly three possible ordered pairs (a,c), namely (1, 4), (2, 2) and (4, 1). There are only two possible ordered pairs (b,d), namely (1, 15) and (3, 5) that need be considered. Thus, there are $3 \cdot 2 = 6$ ordered quadruples of positive integers and an equal number consisting of negative integers, for a total of $\boxed{12}$. Since we have included reversals of the ordered pairs (a,c), considering reversals of the ordered pairs (b,d) would result in duplicates. Thus, there are at most 12 different k-values $(ac+bd)$. If we show that no two distinct ordered quadruples (a,b,c,d) produce the same k-value, then we have exactly 12 different k-values. For $abcd = 60$, a list of the possibilities suffices.

3. $\log_{10}\frac{1}{2} + \log_{10}\frac{1}{3} + \ldots + \log_{10}\frac{1}{n} = \log_{10}1 - \log_{10}2 + \log_{10}1 - \log_{10}3 + \ldots + \log_{10}1 - \log_{10}n$

Since $\log_{10}1 = 0$, we have $-(\log_{10}2 + \log_{10}3 + \ldots + \log_{10}n) = -\log_{10}(n!) = \log_{10}\frac{1}{n!}$. Since $\log_{10}10^{-4} = -4$, we want $\frac{1}{n!} < \frac{1}{10{,}000}$. Since 7! = 5040 and 8! = 40,320, then $\boxed{n = 8}$.

Round 5 – Analytic Geometry

1. Let the y-intercept equal b. Using the intercept form for the equation of a line we have $\frac{x}{4b} + \frac{y}{b} = 1$. If (4, 5) lies on the line, we have $\frac{4}{4b} + \frac{5}{b} = 1 \rightarrow \frac{1}{b} + \frac{5}{b} = 1$ so $b = 6$.

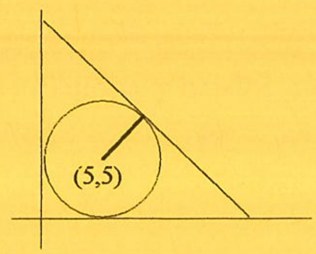
The equation of the line is $\frac{x}{24} + \frac{y}{6} = 1$ and when $y = 18$, $\frac{x}{24} = -2$, so $\boxed{x = -48}$.

Alternate Solution:

Consider the equation: $mx + y = 4m + 5$ as the equation of a line through $P(4, 5)$. The x-intercept is $\frac{4m+5}{m}$ and the y – intercept is $4m + 5$. Solve $\frac{4m+5}{m} = 16m + 20$. This gives $16m^2 + 16m - 5 = 0 \rightarrow (4m - 1)(4m + 5) = 0$. We want $m > 0$ for a negative slope, so $m = \frac{1}{4}$. The equation of the line is $x + 4y = 24$, When $y = 18$, $x = -72 + 24 = -48$.

2. The center of the circle is the same as the center of the hyperbola, namely $(5,3)$. For the hyperbola $a = 3$, so that's the radius of the circle. The equation of the circle is $(x - 5)^2 + (y - 3)^2 = 9$. At $x = 4$, $1 + (y - 3)^2 = 9 \rightarrow (y - 3)^2 = 8 \rightarrow$ $y - 3 = \pm\sqrt{8} \rightarrow y = 3 \pm \sqrt{8}$. The length of the chord is $(3 + \sqrt{8}) - (3 - \sqrt{8}) = \boxed{4\sqrt{2}}$

3. Find the intersection of the radius perpendicular to the tangent with the tangent. The slope of the radius is $\frac{3}{4}$; the equation of the radius is $y - 5 = \frac{3}{4}(x - 5) \rightarrow y = \frac{3}{4}x + \frac{5}{4}$. Substituting



into $(x-5)^2 + (y-5)^2 = 25$ gives $(x-5)^2 + \left(\frac{3}{4}(x-5)\right)^2 = 25$. This simplifies to $\frac{25}{16}(x-5)^2 = 25 \rightarrow x - 5 = \pm 4$. Here x must equal 9, making $y = \frac{3}{4} \cdot 9 + \frac{5}{4} = 8$. Thus, the tangent line passes through (9, 8) with slope $-\frac{4}{3}$. Its equation is $y - 8 = -\frac{4}{3}(x - 9)$, giving $y = -\frac{4}{3}x + 20$. Answer: $\boxed{20}$.

Alternate Solution: Line L: $4x + 3y = 35$ has slope $-\frac{4}{3}$ and passes through the center (5, 5) of the circle, We want a line parallel to L and $r = 5$ units distance from L. Solve $\frac{|C - 35|}{5} = 5$ to get $C = 60$ as the larger solution. The tangent line is then $4x + 3y = 60$ with y-intercept 20.

Round 6 – Trig and Complex Numbers

1. $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}=\dfrac{\dfrac{a}{b}-\dfrac{b+a}{b-a}}{1+\dfrac{a}{b}\cdot\dfrac{b+a}{b-a}}=\dfrac{ab-a^2-b^2-ab}{b^2-ab+ab+a^2}=\dfrac{-\left(a^2+b^2\right)}{a^2+b^2}=\boxed{-1}$.

Solution #2:
Suppose $a=b$. Then $\tan\beta$ is undefined and we have $\beta=90°$, so the triangle must be an isosceles right triangle $\Rightarrow \alpha=45°$. $\tan(\alpha-\beta)=\tan(45°-90°)=-\tan(45°)=\boxed{-1}$

2. The area of ΔABC is $\frac{1}{2}\cdot AB\cdot AC\cdot\sin\angle BAC$. $AB=\sqrt{\sin^2 a+\cos^2 a}=1$. Likewise, $AC=1$.

Thus, $\frac{1}{10}=\frac{1}{2}\cdot1\cdot1\cdot\sin\angle BAC\rightarrow\sin\angle BAC=\frac{1}{5}$. Then $\cos\angle BAC=\frac{\sqrt{24}}{5}=\frac{2\sqrt{6}}{5}$, making

$\tan\angle BAC=\dfrac{1}{2\sqrt{6}}=\boxed{\dfrac{\sqrt{6}}{12}}$.

3. Using the Law of Cosines to compute the distance gives

$MN=\sqrt{a^2+b^2-2ab\cos(83-23)}=\sqrt{a^2+b^2-2ab\cos60}=\sqrt{a^2+b^2-}$

We note that integer values for (a, b, MN) include (3, 8, 7), (5, 8, 7), (7, 15, 13), (8, 15, 13) as indicated below for (3, 8, 7):

$\sqrt{8^2+3^2-8\cdot3}=\sqrt{73-24}=\sqrt{49}=7$. The minimum sum is $8+3=\boxed{11}$.

We need to show that all sums less than 11 fail.

a+b	(a,b)	a^2+b^2-ab	a+b	(a,b)	a^2+b^2-ab
3	(2,1)	5-2=3	9	(8,1)	65-8=57
4	(3,1)	10-3=7		(7,2)	53-14=39
5	(4,1)	17-4=13		(6,3)	45-18=27
	(3,2)	13-6=7		(5,4)	41-20=21
6	(5,1)	26=5=21	10	(9,1)	82-9=73
	(4,2)	20-8=12		(8,2)	68-16=52
7	(6,1)	37-6=31		(7,3)	58-21=37
	(5,2)	29-10=19		(6,4)	52-24=28
	(4,3)	25-12=13	11	(10,1)	101-10=91
8	(7,1)	50-7=43		(9,2)	85-18=67
	(6,2)	40-12=28		(8,3)	73-24=**<u>49</u>**
	(5,3)	34-15=19			

Team Round

1. Since 143 = 11 • 13, neither m nor n can have a factor of 11 or 13. The number of positive integers less than 143 without a factor of 11 or 13 is 142 – 12 – 10 = 120. So there are 120 values for m and for each value of m there is a value of n equal to 143 – m, making for $\boxed{120}$ ordered pairs (m, n).

2. If $a = 2$ we have $\log_2 16 + \log_{16} 128 = 5.75$. If $a = 4$, we have $\log_4 32 + \log_{32} 256 = \frac{41}{10}$, so 4 seems to be the answer. Can we obtain a result less than 4? Using the AM-GM we have

$$\frac{\frac{\ln(8a)}{\ln a} + \frac{\ln(64a)}{\ln(8a)}}{2} \geq \sqrt{\frac{\ln(8a)}{\ln a} \cdot \frac{\ln(64a)}{\ln(8a)}} = \sqrt{\frac{\ln 64 + \ln a}{\ln a}} = \sqrt{\frac{\ln 64}{\ln a} + 1}.$$

Clearly, $\sqrt{\frac{\ln 64}{\ln a} + 1}$ is least for the largest value of a. Letting $a = 4$ gives $\log_a(8a) + \log_{8a}(64a) \geq 2\sqrt{\frac{\ln 4^3}{\ln 4} + 1}$. Since $2\sqrt{\frac{\ln 4^3}{\ln 4} + 1} = 2\sqrt{3+1} = 4$, $\log_a(8a) + \log_{8a}(64a)$ can't drop below 4, so the least possible value of $\lfloor \log_a(8a) + \log_{8a}(64a) \rfloor$ is $\boxed{4}$.

3. Solution #1: Let the number be $ABCD$. Let's count the number of cases where no two adjacent digits are equal. Since none of digits can be 0, there are 9 choices for A. B must be different from A so there are 8 choices for B. C must be different from B but it could equal A so there are 8 choices for C. D must be different from C but it could equal either A or B so there are 8 choices for D. Thus, there are $9 \cdot 8 \cdot 8 \cdot 8$ cases where no two adjacent numbers are the same. There are 9^4 four digit numbers if 0 is not used so the probability that no two adjacent numbers are equal is $\frac{9 \cdot 8^3}{9^4} = \frac{8^3}{9^3}$. Thus, the probability that at least two adjacent numbers are the same is $1 - \frac{8^3}{9^3} = \boxed{\frac{217}{729}}$.

Solution #2: We'll look at all the possible cases where at least 2 adjacent numbers are equal. There are 9^4 possible numbers. Several cases:

1) $AABC$, $BAAC$, or $BCAA$. There are $3 \cdot 9 \cdot 8 \cdot 7$ ways to do that.

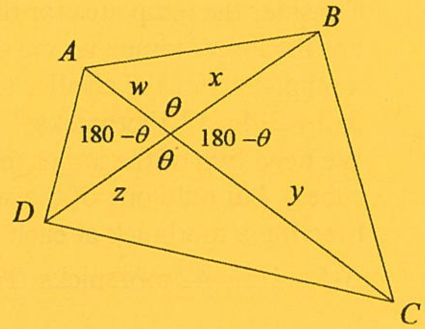
2) $AABB$ or $BBAA$. There are $2 \cdot 9 \cdot 1 \cdot 8 \cdot 1$ such numbers.

3) *AAAB* or *BAAA*. There are $2 \cdot 9 \cdot 1 \cdot 1 \cdot 8$ ways to do that.

4) *AABA* or *ABAA* 16 more.

5) *AAAA* 1 more.

4. Let T be the intersection of diagonals $\overline{AC}$ and $\overline{DB}$. The product of the areas of ΔATB and ΔDTC is $\left(\frac{1}{2}wx\sin\theta\right)\left(\frac{1}{2}yz\sin\theta\right)=\frac{1}{4}xyzw\sin^2\theta$. The product of the areas of ΔATD and ΔBTC is $\left(\frac{1}{2}wz\sin(180-\theta)\right)\left(\frac{1}{2}xy\sin(180-\theta)\right)=\frac{1}{4}xyzw\sin^2\theta$.



Thus, the products of the areas of the opposite triangles are equal: $a(\Delta ATD)\cdot a(\Delta BTC)=a(\Delta ATB)\cdot a(\Delta DTC)$, giving $\frac{a(\Delta ATD)}{a(\Delta ATB)}=\frac{28}{20}=\frac{a(\Delta DTC)}{a(\Delta BTC)}$. Thus, $a(\Delta BTC)=\frac{5}{7}a(\Delta DTC)$. Since $a(\Delta DBC)=120-48=72$, then $a(\Delta DTC)+\frac{5}{7}\left(a(\Delta DTC)\right)=72 \rightarrow a(DTC)=\boxed{42}$.

Alternate Solution 1:

Since the angles at T aren't given, we can assume any angle. Choose 90°. Then choose lengths for TA, TB, and TD that wok for the given areas. Let $TA=8, TB=5, TD=7$. Let $TC=x$. Then $\frac{5x}{2}+\frac{7x}{2}=120-48=72, \rightarrow 12x=144 \rightarrow x=12$. Then area $\Delta DTC=\frac{7\cdot 12}{2}=42$.

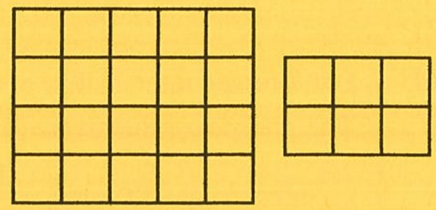
Alternate Solution 2:

Let the area of ΔDTC be X. Then the area of ΔBTC is $120-(20+28)-X=72-X$ Since ΔDTA and ΔBTC have a common altitude from vertex A, their areas are in to ratio of their bases, namely $\frac{DT}{BT}$. The same can be said for ΔDTC and ΔBTC.

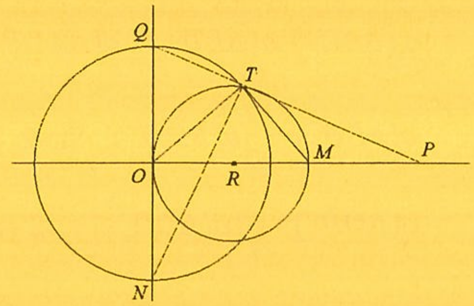
Thus, $\frac{X}{72-X}=\frac{28}{20}=\frac{7}{5}\Rightarrow 12X=7\cdot 72\Rightarrow X=\boxed{42}$

5. Counting the toothpicks on the very bottom gives $5 \cdot 5 + 6 \cdot 4 = 49$ as we count first from front to back and then from side to side. There will be the same number on the top of the first layer. There will be $6 \cdot 5 = 30$ vertical toothpicks in the first row. For the second row, we don't count the toothpicks on the bottom since they have already been counted. We have $3 \cdot 4 = 12$ vertical toothpicks and $3 \cdot 3 + 4 \cdot 2 = 17$ on the top. The total is $49 + 30 + 49 + 12 + 17 = \boxed{157}$.

Alternate Solution:



Consider the templates for the bottoms of each layer. For the larger template, we would need $5 \cdot 5 + 6 \cdot 4 = 49$ toothpicks. For the smaller templates, we would need $3 \cdot 3 + 2 \cdot 4 = 17$ toothpicks.
We need two of the larger templates (for the top and bottom surfaces of the larger layer of cubes), but only one of the smaller templates.
Erecting a toothpick at each lattice point completes the structure, requiring an additional $6 \cdot 5 + 4 \cdot 3 = 42$ toothpicks. Thus, the total need is $2(49) + 17 + 42 = \boxed{157}$

6. 

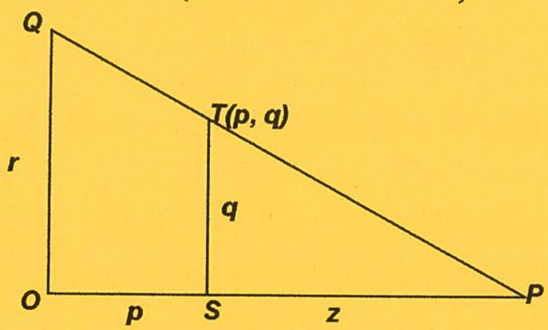
Draw lines $\overline{TO}$, $\overline{TN}$, and $\overline{TM}$ forming right triangles QTN and OTM. Since QTN and QOP are right triangles sharing acute angle $\angle Q$, then $m\angle QNT = m\angle QPO$. Let $m\angle QNT = \alpha$. Since $ON = OT$, then $m\angle OTN = \alpha$. Since $m\angle QTN = m\angle OTM = 90$, we have $m\angle QTO = m\angle NTM = 90 - \alpha$. This means that $m\angle MTP = \alpha$, making ΔTMP isosceles with $TM = MP$. Let $OT = x$ and $TM = MP = y$. Since $OM = 25$ we have $x^2 + y^2 = 25^2 \rightarrow y = \sqrt{625 - x^2}$. For $x = 15$, $y = 20$, making $OP = 25 + 20 = 45$. For $x = 7$, $y = 24$, making $OP = 25 + 24 = 49$. Thus, $45 \le OP \le 49$, making the answer the ordered pair $\boxed{(45, 49)}$.

Alternate Solution:

Let $OQ = r$. Then the equations of the two circles are: $\begin{cases} x^2 + y^2 = r^2 \\ \left(x - \frac{25}{2}\right)^2 + y^2 = \left(\frac{25}{2}\right)^2 \end{cases}$

Subtracting, we have $-25x = -r^2 \Rightarrow x = \left(\frac{r}{5}\right)^2 \Rightarrow y^2 = r^2 - \left(\frac{r}{5}\right)^4 = \frac{r^2\left(625 - r^2\right)}{625} \Rightarrow y = \pm\frac{r}{25}\sqrt{625 - r^2}$

Thus, the coordinates of point T are $\left(\left(\frac{r}{5}\right)^2, +\frac{r}{25}\sqrt{625 - r^2}\right)$



$\frac{z}{z+p} = \frac{q}{r} \Rightarrow z = \frac{pq}{r-q} \Rightarrow OP = p + \frac{pq}{r-q} = \frac{pr}{r-q}$

$r = 15 \Rightarrow T(p,q) = (9,12)$ and we have $OP = \frac{9 \cdot 15}{15 - 12} = 45$

$r = 7 \Rightarrow (p,q) = \left(\frac{49}{25}, \frac{168}{25}\right)$ and we have $OP = \frac{\frac{49}{25} \cdot 7}{7 - \frac{168}{25}} = \frac{49 \cdot 7}{7 \cdot 25 - 168} = \frac{49}{25 - 24} = 49$

Thus, $(a,b) = \boxed{(45,49)}$.