Connecticut State Math Team ARML, 2019

Practice 1
May 11, 2019

Individual Questions

I-1.

For how many positive integers n is $n^3 - 8n^2 + 20n - 13$ a prime number?

I-2.

If a, b, c are real numbers such that $a^2 + 2b = 7$, $b^2 + 4c = -7$, and $c^2 + 6a = -14$, find $a^2 + b^2 + c^2$.

I-3.

Determine the largest positive integer n for which 7^n is a divisor of the integer $\dfrac{200!}{90!30!}$.

I-4.

In quadrilateral $ABCD$, $m\angle B = m\angle C = 120°$, $AB = 3$, $BC = 4$, and $CD = 5$. Find the area of $ABCD$.

I-5.

A triangle has altitudes of length 12, 15, and 20. Compute the degree measure of the largest angle in the triangle.

I-6.

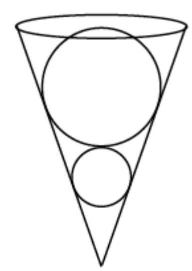
Let a and b be real numbers greater than 1 for which there exists a positive real number c, different from 1, such that

$$2(\log_a c + \log_b c) = 9 \log_{ab} c.$$

Find the largest possible value of $\log_a b$.

I-7.

A right circular cone contains two spheres, as shown. The radius of the larger sphere is 2 times the radius of the smaller sphere. Each sphere is tangent to the other sphere and to the lateral surface of the cone. The larger sphere is tangent to the cone's circular base. Determine the fraction of the cone's volume that is *not* occupied by the two spheres.



I-8.

Let a and b be real numbers such that $\sin a + \sin b = \frac{\sqrt{2}}{2}$ and $\cos a + \cos b = \frac{\sqrt{6}}{2}$. Find $\sin(a + b)$

I-9.

Let a be a fixed real number. Define $M(t)$ to be the maximum value of $-x^2 + 2ax + a$ over all real numbers x with $x \leq t$. Determine a polynomial expression in terms of a that is equal to $M(a-1) + M(a+2)$ for every real number a.

I-10.

In the first few years of the AMC contests a scoring method was used whereby 6 points were given for each correct answer, 2.5 points were given for each unanswered question, and no points were given for an incorrect answer. There were 25 questions on the test. Some of the possible scores between 0 and 150 could be obtained in only one way; for example, the only way to obtain a score of 146.5 was to have 24 correct answers and one unanswered question. Some scores could be obtained in exactly two ways; for example, a score of 104.5 could be obtained with 17 correct answers, 1 unanswered question, and 7 incorrect, and also with 12 correct answers and 13 unanswered questions. There were (three) scores that could be obtained in exactly three ways. What is their sum?

Answers to Individual Questions

I-1. 3

I-2. 14

I-3. 15

I-4. $\frac{47\sqrt{3}}{4}$

I-5. 90

I-6. 2

I-7. $\frac{7}{16}$

I-8. $\frac{\sqrt{3}}{2}$

I-9. $2a^2 + 2a - 1$

I-10. 188.5

Solutions to Individual Questions

I-1.

Because $n^3-8n^2+20n-13=(n-1)(n^2-7n+13)$, for the value to be prime one factor must equal 1 and the other factor must be prime. For $n-1=1$ we must have $n=2$, and in this case the other factor is the prime 3. So $n=2$ is a solution. For $n^2-7n+13=1$, we have $n^2-7n+12=0=(n-4)(n-3)$, so we must have $n=3$ or 4, and in each case the other factor is prime (2 and 3, respectively). Therefore $n^3-8n^2+20n-13$ is a prime for three positive integer values of n.

I-2.

Adding the three equations we obtain

$$(a^2+6a)+(b^2+2b)+(c^2+4c)=-14,$$

which is equivalent to

$$(a+3)^2+(b+1)^2+(c+2)^2=0.$$

Therefore $a=-3$, $b=-1$, $c=-2$, and $a^2+b^2+c^2=14$.

I-3.

First, we count the number of factors of 7 included in 200!.
Every multiple of 7 includes least 1 factor of 7.
The product 200! includes 28 multiples of 7 (since $28\times 7=196$).
Counting one factor of 7 from each of the multiples of 7 (these are $7,14,21,\ldots,182,189,196$), we see that 200! includes at least 28 factors of 7.
However, each multiple of $7^2=49$ includes a second factor of 7 (since $49=7^2, 98=7^2\times 2$, etc.) which was not counted in the previous 28 factors.
The product 200! includes 4 multiples of 49, since $4\times 49=196$, and thus there are at least 4 additional factors of 7 in 200!.
Since $7^3>200$, then 200! does not include any multiples of 7^3 and so we have counted all possible factors of 7.
Thus, 200! includes exactly $28+4=32$ factors of 7, and so $200!=7^{32}\times t$ for some positive

integer t that is not divisible by 7.
Counting in a similar way, the product 90! includes 12 multiples of 7 and 1 multiple of 49, and thus includes 13 factors of 7.
Therefore, $90! = 7^{13} \times r$ for some positive integer r that is not divisible by 7.
Also, 30! includes 4 factors of 7, and thus $30! = 7^4 \times s$ for some positive integer s that is not divisible by 7.
Therefore, $\dfrac{200!}{90!30!} = \dfrac{7^{32} \times t}{(7^{13} \times r)(7^4 \times s)} = \dfrac{7^{32} \times t}{(7^{17} \times rs)} = \dfrac{7^{15} \times t}{rs}$.
Since we are given that $\dfrac{200!}{90!30!}$ is equal to a positive integer, then $\dfrac{7^{15} \times t}{rs}$ is a positive integer.
Since r and s contain no factors of 7 and $7^{15} \times t$ is divisible by rs, then it must be the case that t is divisible by rs.
In other words, we can re-write $\dfrac{200!}{90!30!} = \dfrac{7^{15} \times t}{rs}$ as $\dfrac{200!}{90!30!} = 7^{15} \times \dfrac{t}{rs}$ where $\dfrac{t}{rs}$ is an integer.
Since each of r, s and t does not include any factors of 7, then the integer $\dfrac{t}{rs}$ is not divisible by 7.
Therefore, the largest power of 7 which divides $\dfrac{200!}{90!30!}$ is 7^{15}, and so $n = 15$.

ANSWER: 15

I-4.

Let the extensions of AB beyond B and DC beyond C meet at E. Then, because angles CBE and BCE both equal $60°$, BEC is an equilateral triangle of side 4. The area of triangle BCE is $4\sqrt{3}$ and the area of triangle AED is $\frac{(AE)(DE)\sin 60°}{2} = \frac{63\sqrt{3}}{4}$. Therefore the area of $ABCD$ is $\frac{63\sqrt{3}}{4} - 4\sqrt{3} = \frac{47\sqrt{3}}{4}$.

I-5.

If the sides of a triangle corresponding to the altitudes 12, 15, and 20 are a, b, and c respectively, then we have

$$12a = 15b = 20c$$

or, dividing by 60,

$$\frac{a}{5} = \frac{b}{4} = \frac{c}{3}.$$

It follows that $a = 5r$, $b = 4r$, and $c = 3r$ and so the triangle is a 3-4-5 right triangle whose largest angle is $90°$.

I-6.

Let $x = \log_c a$ and $y = \log_c b$. Then we are given

$$2\left(\frac{1}{x} + \frac{1}{y}\right) = \frac{9}{x+y}.$$

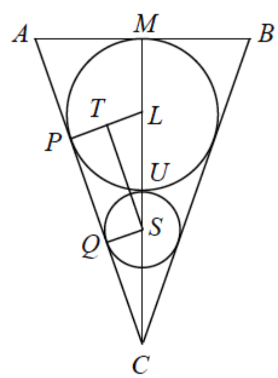
This is equivalent to $2(x+y)^2 = 9xy$ or

$$2x^2 - 5xy + 2y^2 = (2x-y)(x-2y) = 0.$$

Therefore, $2\log_c a = \log_c b$ or $\log_c a = 2\log_c b$, and so $\log_a b = \frac{1}{2}$ or 2. The larger value is 2.

I-7.

Let the radius of the small sphere be r and the radius of the large sphere be $2r$.
Draw a vertical cross-section through the centre of the top face of the cone and its bottom vertex.
By symmetry, this will pass through the centres of the spheres.
In the cross-section, the cone becomes a triangle and the spheres become circles.



We label the vertices of the triangle A, B, C.
We label the centres of the large circle and small circle L and S, respectively.
We label the point where the circles touch U.
We label the midpoint of AB (which represents the centre of the top face of the cone) as M.
Join L and S to the points of tangency of the circles to AC. We call these points P and Q.
Since LP and SQ are radii, then they are perpendicular to the tangent line AC at P and Q,

respectively.
Draw a perpendicular from S to T on LP.
The volume of the cone equals $\frac{1}{3}\pi \cdot AM^2 \cdot MC$. We determine the lengths of AM and MC in terms of r.
Since the radii of the small circle is r, then $QS = US = r$.
Since $TPQS$ has three right angles (at T, P and Q), then it has four right angles, and so is a rectangle.
Therefore, $PT = QS = r$.
Since the radius of the large circle is $2r$, then $PL = UL = ML = 2r$.
Therefore, $TL = PL - PT = 2r - r = r$.
Since MC passes through L and S, it also passes through U, the point of tangency of the two circles.
Therefore, $LS = UL + US = 2r + r = 3r$.
By the Pythagorean Theorem in $\triangle LTS$,

$$TS = \sqrt{LS^2 - TL^2} = \sqrt{(3r)^2 - r^2} = \sqrt{8r^2} = 2\sqrt{2}r$$

since $TS > 0$ and $r > 0$.
Consider $\triangle LTS$ and $\triangle SQC$.
Each is right-angled, $\angle TLS = \angle QSC$ (because LP and QS are parallel), and $TL = QS$.
Therefore, $\triangle LTS$ is congruent to $\triangle SQC$.
Thus, $SC = LS = 3r$ and $QC = TS = 2\sqrt{2}r$.
This tells us that $MC = ML + LS + SC = 2r + 3r + 3r = 8r$.
Also, $\triangle AMC$ is similar to $\triangle SQC$, since each is right-angled and they have a common angle at C.
Therefore, $\dfrac{AM}{MC} = \dfrac{QS}{QC}$ and so $AM = \dfrac{8r \cdot r}{2\sqrt{2}r} = 2\sqrt{2}r$.
This means that the volume of the original cone is $\frac{1}{3}\pi \cdot AM^2 \cdot MC = \frac{1}{3}\pi(2\sqrt{2}r)^2(8r) = \frac{64}{3}\pi r^3$.
The volume of the large sphere is $\frac{4}{3}\pi(2r)^3 = \frac{32}{3}\pi r^3$.
The volume of the small sphere is $\frac{4}{3}\pi r^3$.
The volume of the cone not occupied by the spheres is $\frac{64}{3}\pi r^3 - \frac{32}{3}\pi r^3 - \frac{4}{3}\pi r^3 = \frac{28}{3}\pi r^3$.
The fraction of the volume of the cone that this represents is $\dfrac{\frac{28}{3}\pi r^3}{\frac{64}{3}\pi r^3} = \frac{28}{64} = \frac{7}{16}$.

ANSWER: $\frac{7}{16}$

I-8.

Squaring each equation and adding gives

$$(\sin^2 a + \cos^2 a) + (\sin^2 b + \cos^2 b) + 2(\cos a \cos b + \sin a \sin b) = 2$$

or

$$\cos(a - b) = 0.$$

Multiplying the two equations together gives

$$(\sin a \cos b + \sin b \cos a) + (\sin a \cos a + \sin b \cos b) = \frac{\sqrt{3}}{2}$$

or

$$\sin(a + b) + \sin(a + b) \cos(a - b) = \frac{\sqrt{3}}{2}.$$

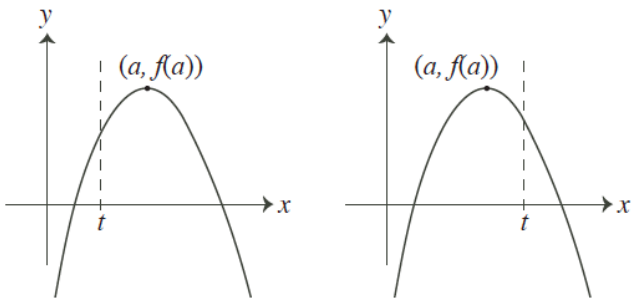
Substituting $\cos(a - b) = 0$, we have $\sin(a + b) = \frac{\sqrt{3}}{2}$.

I-9.

Let $f(x) = -x^2 + 2ax + a$.
Since $(x - a)^2 = x^2 - 2ax + a^2$, then

$$-x^2 + 2ax + a = -(x^2 - 2ax + a^2) + a^2 + a = -(x - a)^2 + a^2 + a$$

Therefore, $M(t)$ is the maximum value of $-(x-a)^2 + a^2 + a$ over all real numbers x with $x \leq t$.
Now the graph of $y = f(x) = -(x - a)^2 + a^2 + a$ is a parabola opening downwards with vertex at $(a, a^2 + a)$.
Since the parabola opens downwards, then the parabola reaches its maximum at the vertex $(a, a^2 + a)$.
Therefore, $f(x)$ is increasing when $x < a$ and decreasing when $x > a$.
This means that, when $t < a$, the maximum of the values of $f(x)$ with $x \leq t$ is $f(t)$ (because $f(x)$ increases until $f(t)$) and when $t \geq a$, the maximum of the values of $f(x)$ with $x \leq t$ is $f(a)$ (because the maximum value of $f(x)$ is to the left of t).



In other words,

$$M(t) = \begin{cases} f(t) & t < a \\ f(a) & t \geq a \end{cases}$$

Since $a - 1 < a$, then $M(a-1) = f(a-1)$.
Since $a + 2 > a$, then $M(a+2) = f(a)$.
Therefore,

$$\begin{aligned} M(a-1) + M(a+2) &= f(a-1) + f(a) \\ &= (-((a-1)-a)^2 + a^2 + a) + (-(a-a)^2 + a^2 + a) \\ &= -1 + a^2 + a - 0 + a^2 + a \\ &= 2a^2 + 2a - 1 \end{aligned}$$

ANSWER: $2a^2 + 2a - 1$

I-10.

If there are c ($c \geq 0$) correct answers and u ($u \geq 0$) unanswered questions and $c + u \leq 25$, then the score is $6c + 2.5u$. If c is sufficiently large and u is sufficiently small, the same score will be obtained with $c - 5$ correct answers and $u + 12$ unanswered questions (this requires $c + u \leq 18$), and also with $c - 10$ correct answers and $u + 24$ unanswered questions. Note that in the latter case we must have $c \geq 10$ and $c + u \leq 11$. Therefore, for there to be three ways to obtain the score $6c + 2.5u$ we can only have $c = 10$ and $u = 0$, or $c = 10$ and $u = 1$, or $c = 11$ and $u = 0$. The three such scores are 60, 62.5, and 66, and their sum is 188.5

CT ARML 2019 Practice #1
Relay Questions

R1-1 Points A, B, C, and D are on the arc of a semicircle, and quadrilateral $ABCD$ is drawn. Compute the numerical value of $\tan A + \tan B + \tan C + \tan D$. Pass back your answer.

R1-2 Let T be TNYWR, and $H = T + 6$. Compute the area of the triangle bounded by the graphs of $y = |2x|$ and $y = H$. Pass back your answer.

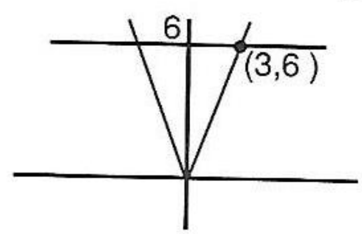
R1-3 Let T be TNYWR, and let $H = \dfrac{T}{18}$. Given right triangle ABC, with hypotenuse $AB = \sqrt{13}$, and $AC = H$. Let D be a point on $\overline{BC}$ such that the area of triangle ABD equals the area of triangle ADC.

If the $m\angle DAC = \dfrac{\pi}{k}$ (in radians), compute k. Turn in k.

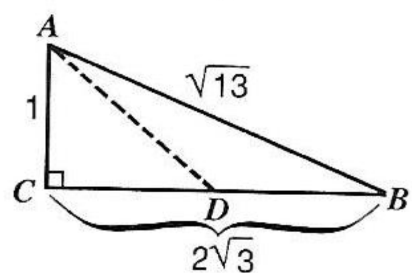
Solutions to Relay 1 (NYSML 1984)

R2-1. No angle can be 90°, and since the opposite angles are supplementary, the sum of the tangents is 0.

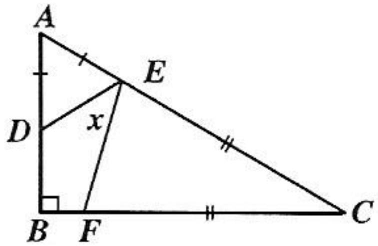
R2-2. $H = 6$. As shown at the right, the area is 18.



R2-3. $H = 1$. Since D must be the midpoint of $\overline{BC}$, $CD = \sqrt{3}$ and $m\angle A = 60° = \pi/3$, so $k = 3$.



R2-1 In right triangle ABC, $\overline{AD} \cong \overline{AE}$ and $\overline{CF} \cong \overline{CE}$ as shown. If $m\angle DEF = x°$ $m\angle DEF = x°$, compute x. Pass back x.



R2-2 Let x be TNYWR. In acute triangle ABC, $AB = BC$.
The altitude from C meets $\overline{AB}$ at F. If $AF : FB = x : 15$,
compute $\cos B$. Pass back your answer.

R2-3 Let t be TNYWR, and let $p = 4t$. The perimeter of a rectangle is p.
A new rectangle is formed whose sides are each x units longer $[x > 0]$ than those of the original rectangle. If the new area is x $units^2$ more than the original area, compute x. Pass in your answer.

Solutions to Relay 2 (ARML 1985)

R1-1. Let angle $AED = p$ and angle $CEF = q$. Then $180 = p + q + x = \frac{180 - A}{2} + \frac{180 - C}{2} + x = 180 + x - \frac{A + C}{2} = 180 + x - 45 \rightarrow x = 45$.

R1-2. $x = 45$. Let $AF = xk$ and $FB = 15k$; then $BC = BA = (15 + x)k$.
Now $\cos B = \frac{15k}{(15 + x)k} = \frac{15}{15 + x} = \frac{15}{60} = \frac{1}{4}$.

R1-3. $t = \frac{1}{4}$, and $p = 1$. Let the dimensions of the original rectangle be a and b.
Then $(a + x)(b + x) = ab + x \rightarrow x(a + b) + x^2 = x \rightarrow a + b + x = 1$.
Since $a + b = \frac{p}{2}$, $x = 1 - \frac{p}{2} = 1 - \frac{1}{2} = \frac{1}{2}$ or .5.

R3-1 Suppose that neither of the three-digit numbers
$M = \underline{4}\ \underline{A}\ \underline{6}$ and $N = \underline{1}\ \underline{B}\ \underline{7}$ is divisible by 9, but the product $M \cdot N$ is divisible by 9.
Compute the largest value of $A + B$.
Pass back your answer.

R3-2 Let T = *TNYWR*. Each interior angle of a rectangle T-gon Has measure $d°$. Compute d. Pass back your answer.

R3-3 Let T = *TNYWR*, and let k be the sum of the distinct prime factors of T. Suppose that r and s are the two roots of the equation $F_k x^2 + F_{k+1}x + F_{k+2} = 0$ where F_n denotes the n^{th} Fibonacci number. Compute the value of $(r+1)(s+1)$. Hand in your answer.

Solutions to Relay 3 (ARML Super relay 2011)

1. In order for the conditions of the problem to be satisfied, M and N must both be divisible by 3, but not by 9. Thus the largest possible value of A is 5, and the largest possible value of B is 7, so $A + B = \mathbf{12}$.

2. From the angle sum formula, $d° = \frac{180°\cdot(T-2)}{T}$. With $T = 12$, $d = \mathbf{150}$.

3. Distributing, $(r+1)(s+1) = rs+(r+s)+1 = \frac{F_{k+2}}{F_k} + \left(-\frac{F_{k+1}}{F_k}\right) + 1 = \frac{F_{k+2}-F_{k+1}}{F_k} + 1 = \frac{F_k}{F_k} + 1 = \mathbf{2}$.

R4-1 Let S be a set of nine distinct integers. Six of the elements of the set are 1, 2, 3, 5. 8 and 13. Compute the number of possible values for the median of S. Pass back your answer.
Pass back your answer.

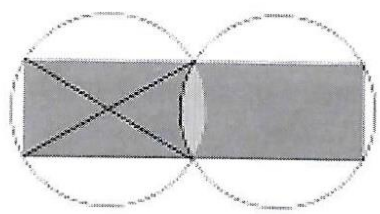
R4-2 Let $T = TNYWR$. Compute the largest integer K such that a $1 \times K$ rectangle can be completely covered by an arrangement of T disks of radius 1. Pass back your answer.

R4-3 Let $T = TNYWR$. ABC and BAD are congruent 30-60-90 triangles, Both with AB as their hypotenuse of length T. If C and D are distinct points in the plane and the intersection of the triangles has positive area, compute the area of the region common to both ABC and BAD.
Hand in your answer.

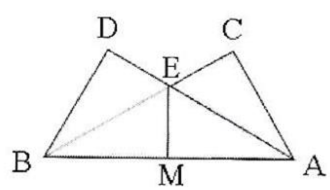
Solutions to Relay 4 (Local ARML 2011)

R2-1. If all of the other elements of S are greater than 8, then the median is 8. If all of the other elements are less than 2, then the median is 2. No three integers exist such that 1, 13, or any other integer less than 2 or greater than 8 can be the median. However, any integer between 2 and 8 (inclusive) can be the median of S, so the answer is $\boxed{7}$.

R2-2. The largest $1 \times n$ rectangle you can cover with a single disk of radius 1 is when $n = \sqrt{3}$. In that case, the center of the rectangle is identical to the center of the circle, and the four corners lie on the circle itself. One can quickly see that T disks can cover at most a $1 \times T\sqrt{3}$ rectangle. As $T = 7$, the answer is $\lfloor 7\sqrt{3} \rfloor = \boxed{12}$. Note, if unable to estimate $\sqrt{3} \approx 1.73$, alternately, consider $\left(7\sqrt{3}\right)^2 = 147$, which is slightly larger than 12^2.



R2-3. Let E be the intersection of AD and BC, and let M be the midpoint of AB. Triangles EBM and EAM are both $30-60-90$ triangles with bases of length $\frac{T}{2}$ and heights of $\frac{T\sqrt{3}}{6}$. The area of triangle ABE is therefore $\frac{T^2\sqrt{3}}{12} = \frac{144\sqrt{3}}{12} = \boxed{12\sqrt{3}}$.



Team Questions
At ARML the TEAM round consists of 10 questions and the team has 20 minutes to finish all of the questions.

CT ARML practice I (15 minutes)

T-1: There are two times between noon and 1pm where the hour and minute hands of a clock are perpendicular. Compute the number of minutes between these two times.

T-2: The three distinct integer roots of $x^3 + qx^2 - 2qx - 8 = 0$ form an arithmetic progression. Compute q.

T-3: Suppose that n leaves a remainder of 24 when divided by 77. If n leaves a remainder of A when divided by 7 and a remainder of B when divided by 11, compute $A + B$.

T-4: Compute the length of the interval of values x for which $\dfrac{1}{x+\sqrt{x}} + \dfrac{1}{x-\sqrt{x}} \geq 1$.

T-5: Compute the number of positive integer factors of $N = 1{,}004{,}006{,}004{,}001$.

T-6: For a positive integer n, let $p(n)$ be the product of the digits of n. Compute the number of positive integers less than 1000 for which $p(p(n)) = 6$.

CT ARML practice (15 minutes) Part II

T-1. In a "Tribonacci" Sequence, each term after the third term is the sum of the three terms immediately preceding it. If the fifth, sixth and seventh terms of a Tribonacci Sequence are 86, 158, and 291 respectively, compute the value of the first term.

T-2. The sum of N consecutive integers is 169, where N is an odd integer such that $1 < N < 169$. Compute N.

T-3. A fraction of the form $1/n$, where n is a positive integer, is called a *unit fraction.* Find 2 unequal unit fractions whose sum is 1/5.

T-4. Quadrilateral $ABCD$ is inscribed in a circle. If $AB = 4$, $BC = 7$, $CD = 8$, and $\overline{AB} \perp \overline{BC}$, compute the length of chord $\overline{AD}$.

T-5. The length of a leg of an isosceles triangle is represented by $x + 2$, and the length of the base by $3x - 4$. Compute all possible values of x.

CT ARML practice (15 minutes)   Part III

T-6. Diagonals $\overline{AC}$ and $\overline{BD}$ of rectangle $ABCD$ intersect at P, and circle O is inscribed in ΔPCD. If $AD = 6$ and $AB = 8$, compute the length of the radius of circle O.

T-7. Compute all values of x such that $x! + \dfrac{24}{x!} = 25$.

T-8. Compute $w + x + y + z$, if

$$\begin{aligned} aw + bx + cy + dz &= a + c \\ bw + cx + dy + az &= b + d \\ cw + dx + ay + bz &= a + b \\ dw + ax + by + cz &= c + d \end{aligned}$$

and $a + b \neq -(c + d)$.

T-9. Compute θ for $180° < \theta < 360°$, such that $(\tan\theta + \cot\theta = 4)$.

T-10. If $\log_6 2 = a$, express $\log_6 0.375$ in terms of a.

Team Round

T-1: There are two times between noon and 1pm where the hour and minute hands of a clock are perpendicular. Compute the number of minutes between these two times.

Answer: 360/11

Solution: The minute hand moves at a rate of 360 degrees per hour (or 6 degrees per minute), the hour hand moves at a rate of 30 degrees per hour (or 0.5 degrees per minute). Therefore, the angle between the hour and the minute hand increases by 5.5 degrees per minute. The first time the hands are perpendicular, the angle between them is 90 degrees. The second time is when the angle is 270 degrees, which occurs 180/5.5 or $\boxed{360/11}$ minutes later.

T-2: The three distinct integer roots of $x^3+qx^2-2qx-8=0$ form an arithmetic progression. Compute q.

Answer: 3

Solution: The product of the roots is -8, and the possible sets of distinct integer roots are $\{\pm 1,\pm 2,\pm 4\}$, $\{\pm 2,\pm 2,\pm 2\}$, and $\{\pm 1,\pm 1,\pm 8\}$ where either zero or two of the roots are negative. Of all of the cases listed here, only one has distinct roots in arithmetic progression with two roots negative, $\{-4,-1,2\}$ The corresponding polynomial is $(x+2)(x-1)(x-4)=x^3+3x^2-6x-8$, and the answer is $q=\boxed{3}$.

T-3: Suppose that n leaves a remainder of 24 when divided by 77. If n leaves a remainder of A when divided by 7 and a remainder of B when divided by 11, compute $A+B$.

Answer: 5

Solution: $n=77k+24$ for some integer value of k. When n is divided by 7, the $77k$ term divides evenly, so the remainder is the residue of 24 modulo 7, or 3. Similarly when n is divided by 11, the remainder is 2. The answer is 3+2=$\boxed{5}$.

T-4: Compute the length of the interval of values x for which $\dfrac{1}{x+\sqrt{x}}+\dfrac{1}{x-\sqrt{x}}\geq 1$.

Answer: 2

Solution: $\dfrac{1}{x+\sqrt{x}}+\dfrac{1}{x-\sqrt{x}}=\dfrac{2x}{x(x-1)}$. The domain of the function is all positive x except 1. For x between 0 and 1, the denominator is negative, so the function is negative. For x greater than 1, the function is strictly decreasing. It equals 1 when $\frac{2x}{x(x-1)} = 1 \to 2x = x^2 - x \to x^2 - 3x = 0 \to x = 3$, so the interval is (1,3), which has length $\boxed{2}$.

T-5: Compute the number of positive integer factors of N = 1,004,006,004,001.

Answer: 125

Solution: Considering that the non-zero digits of N look like binomial coefficients, N is the binomial expansion of $(1000+1)^4$. Since $1001 = 7 \times 11 \times 13$, $1{,}004{,}006{,}004{,}001 = 7^4 \times 11^4 \times 13^4$. The prime factorization of any factor of N will have 0 to 4 powers of each of the primes 7, 11, and 13, so the total number of factors is 5^3 = $\boxed{125}$.

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T-1. 8. Let the sequence be a,b,c,d,86,158,291. By definition, $d + 86 + 158 = 291 \Rightarrow d=47$. Next, $c + 47 + 86 = 158 \Rightarrow c=25$. Then, $b + 25 + 47 = 86 \Rightarrow b=14$. Finally, $a + 14 + 25 = 47 \Rightarrow a=8$.

T-2. 13. Let x be the median of the consecutive integers. Since N is odd, the sum can be represented by $(x – a)+(x – a+1)+\ldots+x+\ldots+(x+a – 1)+(x+a)$. Summing gives $N_x=169$. N must be 13 because 13 is the only factor of 169 between 1 and 169.

T-3. $\frac{1}{6}, \frac{1}{30}$. Since $\frac{n+1}{x}=\frac{n}{x}+\frac{1}{x}$, if both n and n+1 are factors of x, all three fractions will reduce to unit fractions. Let n=5, n+1=6, and x=30. This gives $\frac{6}{30}=\frac{5}{30}+\frac{1}{30}, or \frac{1}{5}=\frac{1}{6}+\frac{1}{30}$.

T-4. 1. Draw $\overline{AC}$. Since ΔABC is a right triangle, $\overline{AC}$ is a diameter, and ΔADC is also a right triangle. Therefore $AC^2=AB^2+BC^2=AD^2+CD^2$, or $4^2+7^2=AD^2+8^2 \Rightarrow AD = 1$.

T-5. 4/3< x <8. By the triangle inequality, $2(x+2)>3x – 4$, leading to $x<8$. In addition, x must satisfy $(3x – 4)+(x+2)=(x+2)$. This leads to $x >4/3$.

T-6. 4/3. $PC = PD = 5$. Draw $\overline{OM} \perp \overline{PC}$. Let $OM = r$ so that $OP = 3 - r$. $\Delta OMP \sim \Delta ADC \Rightarrow \frac{r}{3-r}=\frac{8}{10} \Rightarrow 10r = 24 – 8r \Rightarrow r = \frac{4}{3}$.

T-7. 0, 1, 4. We have $x!+\frac{24}{x!} = 25 \Rightarrow (x!)^2+24 = 25(x!)^2 – 25(x!) + 24 = 0 \Rightarrow (x! – 24)(x! – 1)=0$. Therefore $x! = 24$ with $x = 4$, or $x! = 1$ with $x = 0$ or $x = 1$.

T-8. 2. Adding the four equations gives $(a+b+c+d)w + (a+b+c+d)x + (a+b+c +d)y + (a+b+c+d)z = 2a+2b+2c+2d$ or $(a+b+c+d)(w+x+y+z)= 2(a+b+c+d) \Rightarrow w+x+y+z = 2$.

T-9. 195°, 255°. The equation $\tan\theta + \cot\theta = 4 \Rightarrow \frac{\sin 6}{\cos\theta}+\frac{\cos 6}{\sin\theta}=4$ $\Rightarrow \frac{\sin^2\theta+\cos^2\theta}{\sin 6 \cos 6}=4 \Rightarrow \frac{1}{\sin 6 \cos 6} = 4 \Rightarrow 2 \sin\theta \cos\theta = ½ \Rightarrow \sin 2\theta = ½$. Therefore $2\theta = 30° \pm 360k°$, $k \in \mathbf{Z}$, or $2\theta = 150° \pm 360k°$, $k \in \mathbf{Z}$. This means $\theta = 15° \pm 180k°$, or $\theta = 75° \pm 180k°$. The only values of θ falling in the correct range are $195° = 15 + 180$ and $255° = 75 + 180$.

T-10. 1 – 4a. Rewriting gives $\log_6 .375 = \log_6 \frac{3}{8} = \log_6 \frac{6}{16} = \log_6 6 – \log_6 16 = 1 – \log_6 2^4 = 1 – 4\log_6 2 = 1 –$ 4a.