Connecticut State
Math Team
ARML, 2019

Practice 2
May 18, 2019

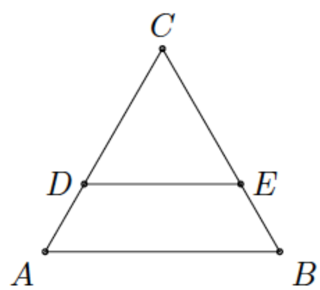
Individual Questions

I-1.

Let p be the largest positive integer such that 2^p divides $13^4 - 11^4$. Compute the value of p.

I-2.

In the equilateral triangle $\triangle ABC$, the point D is on $\overline{AC}$, the point E is on $\overline{BC}$, and $\overline{DE}$ is parallel to $\overline{AB}$. If the perimeter of the triangle $\triangle DEC$ is equal to the perimeter of the trapezoid $ABED$, what is the ratio of the areas of the triangle $\triangle DEC$ and the trapezoid $ABED$?



(Give your answer as a number, which might be a fraction. Do not use the colon ratio notation.)

I-3.

Consider the set of all fractions x/y where x and y are relatively prime positive integers. How many of these fractions have the property that if both the numerator and the denominator are increased by 1, the value of the fraction is increased by 10%?

I-4.

For each positive integer n, the parabola $y = (n^2 + n)x^2 - (2n + 1)x + 1$ intersects the x-axis at the points A_n and B_n. Let $S(m) = A_1B_1 + A_2B_2 + \cdots + A_mB_m$. Compute $S(1000)$.

I-5.

Tom has 12 coins, each of which is a nickel or a dime. There are exactly 17 different values that can be obtained as combinations of one or more of his coins. How many dimes does Tom have?

I-6.

For certain real numbers a, b, and c, the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

has three distinct zeros, and each zero of $g(x)$ is also a zero of the polynomial

$$f(x) = x^4 + x^3 + bx^2 + 100x + c.$$

What is $f(1)$?

I-7.

A regular 6-sided die with the numbers 1, 2, 3, 4, 5 and 6 is rolled twice. Next, another regular 6-sided die is rolled two times. What is the probability that the sum of the numbers rolled on the first 6-sided die is greater than the sum of the numbers rolled on the second 6-sided die?

I-8.

Suppose s and t are integers satisfying $s\sqrt{s-t} = t\sqrt{2t+s}$. If $5 \leq s \leq 20$ and $5 \leq t \leq 20$, then how many such solutions are there?

I-9.

Suppose x, y and z are three real numbers such that $3x$, $4y$, $5z$ is a geometric sequence and $1/x$, $1/y$, $1/z$ is an arithmetic sequence. What is the value of $(x/z) + (z/x)$?

I-10.

How many different ways can one order the integers 1 through 5 so that no three consecutive integers in the ordering are in increasing order?

Answers to Individual Questions

I-1. 5

I-2. $\frac{9}{7}$

I-3. 1

I-4. $\frac{1000}{1001}$

I-5. 5

I-6. -7007

I-7. $\frac{575}{1296}$

I-8. 6

I-9. $\frac{34}{15}$

I-10. 70

Solutions to Individual Questions

I-1.

Answer: 5

Solution: We have $13^4 - 11^4 = (13^2)^2 - (11^2)^2 = (13^2 - 11^2)(13^2 + 11^2) = 48 \cdot 290 = 16 \cdot 3 \cdot 2 \cdot 145$. $\square$

I-2.

Answer: $\frac{9}{7}$

Solution: Let $AB = a$ and $CD = x$. Clearly, $\triangle CDE$ is equilateral and its perimeter is $3x$. The perimeter of the trapezoid $ABED$ is $a+(a-x)+x+(a-x) = 3a-x$. Thus, $3x = 3a-x$, or $x = 3a/4$. Therefore, the ratio of the areas of $\triangle DEC$ and $\triangle ABC$ is $9/16$. $\square$

I-3.

Answer: 1

Solution: We have $1.1x/y = (x+1)/(y+1)$ or $1.1x(y+1) = y(x+1)$. Multiplying by 10 and simplifying leads to $xy + 11x - 10y = 0$ or $11x = y(10-x)$. Thus, y divides $11x$. Since x and y are relatively prime, y divides 11, so $y = 1$ or $y = 11$. If $y = 1$, then $x = 5/6$, not an integer. If $y = 11$, $x = 5$. $\square$

I-4.

Answer: $\frac{1000}{1001}$

Solution: The zeros of the quadratic equation $(n^2+n)x^2 - (2n+1)x + 1 = 0$ are $1/n$ and $1/(n+1)$. Thus,

$$A_nB_n = \frac{1}{n} - \frac{1}{n+1}.$$

So $S(m) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{m} - \frac{1}{m+1}\right) = 1 - \frac{1}{m+1} = \frac{m}{m+1}$, making $S(1000) = \frac{1000}{1001}$.

I-5.

Answer: 5

Solution: Clearly, Tom has at least one nickel (otherwise there are exactly 12 values one can make with his coins). Thus, the values one can make are all consecutive multiples of 5 cents, starting with 5 cents and ending with the total value of Tom's coins. Since there are 17 values one can make, the total value of Tom's coins is 85 cents. If Tom has n nickels and d dimes we get the system of equations $n + d = 12$ and $5n + 10d = 17$. Solving the system we get $n = 7$ and $d = 5$. $\square$

I-6.

Answer: -7007

Solution: Since g has distinct zeros and each zero of $g(x)$ is a zero of $f(x)$, then $g(x)$ divides $f(x)$. Write $f(x) = q(x) \cdot g(x) + r(x)$ with quotient $q(x)$ and remainder $r(x)$ given by

$$q(x) = x + (1-a), \qquad r(x) = \big(b - 1 - a(1-a)\big)x^2 + \big(90 - (1-a)\big)x + \big(c - 10(1-a)\big).$$

Thus, $b-1-a(1-a) = 0$, $90-(1-a) = 0$, and $c-10(1-a)$. Therefore, $a = -89$, $b = -8009$, and $c = 900$. □

I-7.

Answer: $\frac{575}{1296}$

Solution: Let S_1 be the sum of the numbers rolled on the first 6-sided die and let S_2 be the sum of the numbers rolled on the second 6-sided die. Let p_1 be the probability that $S_1 > S_2$, let p_2 be the probability that $S_1 = S_2$, and let p_3 be the probability that $S_1 < S_2$. Clearly, $p_1 + p_2 + p_3 = 1$ and $p_1 = p_3$. Therefore, $p_1 = (1/2)(1 - p_2)$. S_1 can be any integer from 2 to 12. The probability that $S_1 = k$ for k between 2 and 7 is $(k-1)/36$; and the probability that $S_1 = k$ for k between 8 and 12 is $(13-k)/36$. Thus, the probability that $S_1 = S_2 = k$ is $(k-1)^2/(36)^2$ when k is between 2 and 7 ; and the probability that $S_1 = S_2 = k$ is $(13-k)^2/(36)^2$ when k is between 8 and 12. Therefore, $p_2 = 146/1296$. □

I-8.

Answer: 6

Solution: If we square the given equation we get $s^2(s-t) = t^2(2t+s)$ or $s^3 - s^2t - st^2 - 2t^3 = 0$. This factors as $(s-2t)(s^2+st+t^2) = 0$. Now $s^2 + st + t^2 = 0$ only if $s = t = 0$. Indeed, $s^2 + st + t^2 = (s + t/2)^2 + (3/4)t^2$. Thus, the equation holds only if $s = 2t$ and $s \geq 0$. Furthermore, s and t are in the given ranges exactly when $t = 5, 6, 7, 8, 9$, or 10. □

I-9.

Answer: $\frac{34}{15}$

Solution: Since, $3x$, $4y$, $5z$ is a geometric sequence, $(4y)/(3x) = (5z)/(4y)$, so $16y^2 = 15xz$. Also, since $1/x, 1/y, 1/z$ is an arithmetic sequence,

$$\frac{1}{y} - \frac{1}{x} = \frac{1}{z} - \frac{1}{z},$$

or

$$\frac{2}{y} = \frac{1}{x} + \frac{1}{z} = \frac{x+z}{xz}.$$

On the other hand,

$$\frac{x}{z} + \frac{z}{x} = \frac{x^2+z^2}{xz} = \frac{(x+z)^2}{xz} - 2 = \frac{4xz}{y^2} - 2 = \frac{64}{15} - 2.$$

□

I-10.

Answer: 70

Solution: Let A_1 be the set of all permutations $(a_1, a_2, a_3, a_4, a_5)$ of the integers $1, 2, 3, 4, 5$ such that $a_1 < a_2 < a_3$; let A_2 be the set of all permutations $(a_1, a_2, a_3, a_4, a_5)$ of $1, 2, 3, 4, 5$ such that $a_2 < a_3 < a_4$; and let A_3 be the set of all permutations $(a_1, a_2, a_3, a_4, a_5)$ of the integers $1, 2, 3, 4, 5$ such that $a_3 < a_4 < a_5$. We need to count how many permutations of $1, 2, 3, 4, 5$ are not in the union of the sets A_1, A_2, and A_3.

First, there are $5! = 120$ permutations of $1, 2, 3, 4, 5$.

Next, by inclusion-exclusion

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

We have $|A_1| = |A_2| = |A_3| = \binom{5}{3} \cdot 2 = 20$ (for example, to compute $|A_1|$ note that there are $\binom{5}{3}$ ways to choose a_1, a_2, and a_3, and two ways to pick a_4 out of the remaining two numbers).

Also, $A_1 \cap A_2$ is the set of all permutations of $1, 2, 3, 4, 5$ where $a_1 < a_2 < a_3 < a_4$, there are 5 such permutations (there are 5 choices for a_5). Similarly, $A_2 \cap A_3$ is the set of all permutations of $1, 2, 3, 4, 5$ where $a_2 < a_3 < a_4 < a_5$, there are 5 such permutations.

Finally, $A_1 \cap A_3$ and $A_1 \cap A_2 \cap A_3$ is the set of all permutations of $1, 2, 3, 4, 5$ where $a_1 < a_2 < a_3 < a_4 < a_5$, that is the set $\{(1, 2, 3, 4, 5)\}$.

Thus, $|A_1 \cup A_2 \cup A_3| = 20 + 20 + 20 - 5 - 1 - 5 + 1 = 50$. □

ARML Team Questions 2011

T-1. If $1, x, y$ is a geometric sequence and $x, y, 3$ is an arithmetic sequence, compute the maximum value of $x + y$.

T-2. Define the sequence of positive integers $\{a_n\}$ as follows:

$$\begin{cases} a_1 = 1; \\ \text{for } n \geq 2,\ a_n \text{ is the smallest possible positive value of } n - {a_k}^2, \text{ for } 1 \leq k < n. \end{cases}$$

For example, $a_2 = 2 - 1^2 = 1$, and $a_3 = 3 - 1^2 = 2$. Compute $a_1 + a_2 + \cdots + a_{50}$.

T-3. Compute the base b for which $253_b \cdot 341_b = \underline{7}\,\underline{4}\,\underline{X}\,\underline{Y}\,\underline{Z}_b$, for some base-$b$ digits X, Y, Z.

T-4. Some portions of the line $y = 4x$ line below the curve $y = 10\pi \sin^2 x$, and other portions lie above the curve. Compute the sum of the lengths of all the segments of the graph of $y = 4x$ that lie in the first quadrant, below the graph of $y = 10\pi \sin^2 x$.

T-5. In equilateral hexagon $ABCDEF$, $\mathrm{m}\angle A = 2\mathrm{m}\angle C = 2\mathrm{m}\angle E = 5\mathrm{m}\angle D = 10\mathrm{m}\angle B = 10\mathrm{m}\angle F$, and diagonal $BE = 3$. Compute $[ABCDEF]$, that is, the area of $ABCDEF$.

T-6. The *taxicab distance* between points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ is defined as $d(A, B) = |x_A - x_B| + |y_A - y_B|$. Given some $s > 0$ and points $A = (x_A, y_A)$ and $B = (x_B, y_B)$, define the *taxicab ellipse* with foci $A = (x_A, y_A)$ and $B = (x_B, y_B)$ to be the set of points $\{Q \mid d(A, Q) + d(B, Q) = s\}$. Compute the area enclosed by the taxicab ellipse with foci $(0, 5)$ and $(12, 0)$, passing through $(1, -1)$.

T-7. The function f satisfies the relation $f(n) = f(n-1)f(n-2)$ for all integers n, and $f(n) > 0$ for all positive integers n. If $f(1) = \frac{f(2)}{512}$ and $\frac{1}{f(1)} = 2f(2)$, compute $f(f(4))$.

T-8. Frank Narf accidentally read a degree n polynomial with integer coefficients backwards. That is, he read $a_n x^n + \ldots + a_1 x + a_0$ as $a_0 x^n + \ldots + a_{n-1}x + a_n$. Luckily, the reversed polynomial had the same zeros as the original polynomial. All the reversed polynomial's zeros were real, and also integers. If $1 \leq n \leq 7$, compute the number of such polynomials such that $\mathrm{GCD}(a_0, a_1, \ldots, a_n) = 1$.

T-9. Given a regular 16-gon, extend three of its sides to form a triangle none of whose vertices lie on the 16-gon itself. Compute the number of noncongruent triangles that can be formed in this manner.

T-10. Two square tiles of area 9 are placed with one directly on top of the other. The top tile is then rotated about its center by an acute angle θ. If the area of the overlapping region is 8, compute $\sin\theta + \cos\theta$.

2011 Team Answers

T-1. $\frac{15}{4}$

T-2. 253

T-3. 20

T-4. $\frac{5\pi}{4}\sqrt{17}$

T-5. $\frac{9}{2}$

T-6. 96

T-7. 4096

T-8. 70

T-9. 11

T-10. $\frac{5}{4}$

6 Team Solutions

Problem 1. If $1, x, y$ is a geometric sequence and $x, y, 3$ is an arithmetic sequence, compute the maximum value of $x + y$.

Solution 1. The common ratio in the geometric sequence $1, x, y$ is $\frac{x}{1} = x$, so $y = x^2$. The arithmetic sequence $x, y, 3$ has a common difference, so $y - x = 3 - y$. Substituting $y = x^2$ in the equation yields

$$\begin{aligned} x^2 - x &= 3 - x^2 \\ 2x^2 - x - 3 &= 0, \end{aligned}$$

from which $x = \frac{3}{2}$ or -1. The respective values of y are $y = x^2 = \frac{9}{4}$ or 1. Thus the possible values of $x + y$ are $\frac{15}{4}$ and 0, so the answer is $\mathbf{\frac{15}{4}}$.

Problem 2. Define the sequence of positive integers $\{a_n\}$ as follows:

$$\begin{cases} a_1 = 1; \\ \text{for } n \geq 2,\ a_n \text{ is the smallest possible positive value of } n - a_k{}^2, \text{ for } 1 \leq k < n. \end{cases}$$

For example, $a_2 = 2 - 1^2 = 1$, and $a_3 = 3 - 1^2 = 2$. Compute $a_1 + a_2 + \cdots + a_{50}$.

Solution 2. The requirement that a_n be the smallest positive value of $n - a_k{}^2$ for $k < n$ is equivalent to determining the largest value of a_k such that $a_k{}^2 < n$. For $n = 3$, use either $a_1 = a_2 = 1$ to find $a_3 = 3 - 1^2 = 2$. For $n = 4$, the strict inequality eliminates a_3, so $a_4 = 4 - 1^2 = 3$, but a_3 can be used to compute $a_5 = 5 - 2^2 = 1$. In fact, until $n = 10$, the largest allowable prior value of a_k is $a_3 = 2$, yielding the values $a_6 = 2$, $a_7 = 3$, $a_8 = 4$, $a_9 = 5$. In general, this pattern continues: from $n = m^2 + 1$ until $n = (m + 1)^2$, the values of a_n increase from 1 to $2m + 1$

Let $S_m = 1 + 2 + \cdots + (2m + 1)$. Then the problem reduces to computing $S_0 + S_1 + \cdots + S_6 + 1$, because $a_{49} = 49 - 6^2$ while $a_{50} = 50 - 7^2 = 1$. $S_m = \frac{(2m+1)(2m+2)}{2} = 2m^2 + 3m + 1$, so

$$\begin{aligned} S_0 + S_1 + S_2 + S_3 + S_4 + S_5 + S_6 &= 1 + 6 + 15 + 28 + 45 + 66 + 91 \\ &= 252. \end{aligned}$$

Therefore the desired sum is $252 + 1 = \mathbf{253}$.

Problem 3. Compute the base b for which $253_b \cdot 341_b = \underline{7}\,\underline{4}\,\underline{X}\,\underline{Y}\,\underline{Z}_b$, for some base-$b$ digits X, Y, Z.

Solution 3. Write $253_b \cdot 341_b = (2b^2 + 5b + 3)(3b^2 + 4b + 1) = 6b^4 + 23b^3 + 31b^2 + 17b + 3$. Compare the coefficients in this polynomial to the digits in the numeral $\underline{7}\,\underline{4}\,\underline{X}\,\underline{Y}\,\underline{Z}$. In the polynomial, the coefficient of b^4 is 6, so there must be a carry from the b^3 place to get the $7b^4$ in the numeral. After the carry, there should be no more than 4 left for the coefficient of b^3 as only one b is carried. Therefore $23 - b \leq 4$ or $b \geq 19$.

By comparing digits, note that $Z = 3$. Then

$$\begin{aligned} 6b^4 + 23b^3 + 31b^2 + 17b &= \underline{7}\,\underline{4}\,\underline{X}\,\underline{Y}\,\underline{0} \\ &= 7b^4 + 4b^3 + X \cdot b^2 + Y \cdot b. \end{aligned}$$

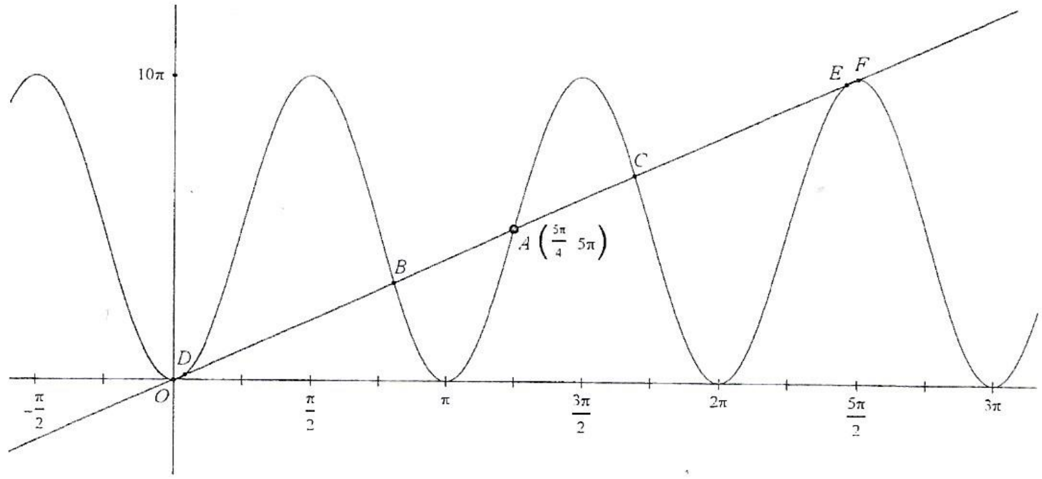
Because $b > 0$, this equation can be simplified to

$$b^3 + X \cdot b + Y = 19b^2 + 31b + 17.$$

Thus $Y = 17$ and $b^2 + X = 19b + 31$, from which $b(b - 19) = 31 - X$. The expression on the left side is positive (because $b > 19$) and the expression on the right side is at most 31 (because $X > 0$), so the only possible solution is $b = 20$, $X = 11$. The answer is **20**.

Problem 4. Some portions of the line $y = 4x$ line below the curve $y = 10\pi\sin^2 x$, and other portions lie above the curve. Compute the sum of the lengths of all the segments of the graph of $y = 4x$ that lie in the first quadrant, below the graph of $y = 10\pi\sin^2 x$.

Solution 4. Notice first that all intersections of the two graphs occur in the interval $0 \le x \le \frac{5\pi}{2}$, because the maximum value of $10\pi\sin^2 x$ is 10π (at odd multiples of $\frac{\pi}{2}$), and $4x > 10\pi$ when $x > \frac{5\pi}{2}$. The graphs are shown below.



Within that interval, both graphs are symmetric about the point $A = \left(\frac{5\pi}{4}, 5\pi\right)$. For the case of $y = 10\pi\sin^2 x$, this symmetry can be seen by using the power-reducing identity $\sin^2 x = \frac{1-\cos 2x}{2}$. Then the equation becomes $y = 5\pi - 5\pi\cos 2x$, which has amplitude 5π about the line $y = 5\pi$, and which crosses the line $y = 5\pi$ for $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$.

Label the points of intersection A, B, C, D, E, F, and O as shown. Then $\overline{AB} \cong \overline{AC}$, $\overline{BD} \cong \overline{CE}$, and $\overline{OD} \cong \overline{EF}$ Thus

$$\begin{aligned} BD + AC + EF &= OD + DB + BA \\ &= OA. \end{aligned}$$

By the Pythagorean Theorem,

$$\begin{aligned} OA &= \sqrt{\left(\frac{5\pi}{4}\right)^2 + (5\pi)^2} \\ &= \frac{5\pi}{4}\sqrt{1^2 + 4^2} \\ &= \frac{5\pi}{4}\sqrt{17}. \end{aligned}$$

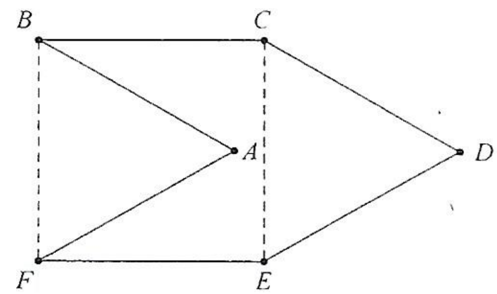
Problem 5. In equilateral hexagon $ABCDEF$, $m\angle A = 2m\angle C = 2m\angle E = 5m\angle D = 10m\angle B = 10m\angle F$, and diagonal $BE = 3$. Compute $[ABCDEF]$, that is, the area of $ABCDEF$.

Solution 5. Let $m\angle B = \alpha$. Then the sum of the measures of the angles in the hexagon is:

$$\begin{aligned} 720^\circ &= m\angle A + m\angle C + m\angle E + m\angle D + m\angle B + m\angle F \\ &= 10\alpha + 5\alpha + 5\alpha + 2\alpha + \alpha + \alpha = 24\alpha. \end{aligned}$$

Thus $30° = \alpha$ and $m\angle A = 300°$, so the exterior angle at A has measure $60° = m\angle D$. Further, because $AB = CD$ and $DE = AF$, it follows that $\triangle CDE \cong \triangle BAF$. Thus

$$[ABCDEF] = [ABCEF] + [CDE] = [ABCEF] + [ABF] = [BCEF].$$



To compute $[BCEF]$, notice that because $m\angle D = 60°$, $\triangle CDE$ is equilateral. In addition,

$$\begin{aligned} 150° &= m\angle BCD \\ &= m\angle BCE + m\angle DCE = m\angle BCE + 60°. \end{aligned}$$

Therefore $m\angle BCE = 90°$. Similarly, because the hexagon is symmetric, $m\angle CEF = 90°$, so quadrilateral $BCEF$ is actually a square with side length 3. Thus $CE = \frac{BE}{\sqrt{2}} = \frac{3}{\sqrt{2}}$, and $[ABCDEF] = [BCEF] = \frac{9}{2}$.

Alternate Solution: Calculate the angles of the hexagon as in the first solution. Then proceed as follows.

First, $ABCDEF$ can be partitioned into four congruent triangles. Because the hexagon is equilateral and $m\angle ABC = m\angle AFE = 30°$, it follows that $\triangle ABC$ and $\triangle AFE$ are congruent isosceles triangles whose base angles measure $75°$. Next, $m\angle ABC + m\angle BCD = 30° + 150° = 180°$, so $\overline{AB} \parallel \overline{CD}$. Because these two segments are also congruent, quadrilateral $ABCD$ is a parallelogram. In particular, $\triangle CDA \cong \triangle ABC$. Similarly, $\triangle EDA \cong \triangle AFE$.

Now let $a = AC = AE$ be the length of the base of these isosceles triangles, and let $b = AB$ be the length of the other sides (or of the equilateral hexagon). Because the four triangles are congruent, $[ABCDEF] = [ABC] + [ACD] + [ADE] + [AEF] = 4[ABC] = 4 \cdot \frac{1}{2} b^2 \sin 30° = b^2$.

Applying the Law of Cosines to $\triangle ABC$ gives $a^2 = b^2 + b^2 - 2b^2 \cos 30° = (2 - \sqrt{3})b^2$. Because $4 - 2\sqrt{3} = (\sqrt{3} - 1)^2$, this gives $a = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right) b$. Using the given length $BE = 3$ and applying the Law of Cosines to $\triangle ABE$ gives

$$\begin{aligned} 9 &= a^2 + b^2 - 2ab \cos 135° \\ &= a^2 + b^2 + \sqrt{2}ab \\ &= (2 - \sqrt{3})b^2 + b^2 + (\sqrt{3} - 1)b^2 \\ &= 2b^2. \end{aligned}$$

Thus $[ABCDEF] = b^2 = \frac{9}{2}$.

Problem 6. The *taxicab distance* between points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ is defined as $d(A, B) = |x_A - x_B| + |y_A - y_B|$. Given some $s > 0$ and points $A = (x_A, y_A)$ and $B = (x_B, y_B)$, define the *taxicab ellipse* with foci $A = (x_A, y_A)$ and $B = (x_B, y_B)$ to be the set of points $\{Q \mid d(A, Q) + d(B, Q) = s\}$. Compute the area enclosed by the taxicab ellipse with foci $(0, 5)$ and $(12, 0)$, passing through $(1, -1)$.

Solution 6. Let $A = (0,5)$ and $B = (12,0)$, and let $C = (1,-1)$. First compute the distance sum: $d(A,C) + d(B,C) = 19$. Notice that if $P = (x,y)$ is on the segment from $(0,-1)$ to $(12,-1)$, then $d(A,P) + d(B,P)$ is constant. This is because if $0 < x < 12$,

$$\begin{aligned} d(A,P) + d(B,P) &= |0-x| + |5-(-1)| + |12-x| + |0-(-1)| \\ &= x + 6 + (12-x) + 1 \\ &= 19. \end{aligned}$$

Similarly, $d(A,P) + d(P,B) = 19$ whenever P is on the segment from $(0,6)$ to $(12,6)$. If P is on the segment from $(13,0)$ to $(13,5)$, then P's coordinates are $(13,y)$, with $0 \le y \le 5$, and thus

$$\begin{aligned} d(A,P) + d(B,P) &= |0-13| + |5-y| + |12-13| + |0-y| \\ &= 13 + (5-y) + 1 + y \\ &= 19. \end{aligned}$$

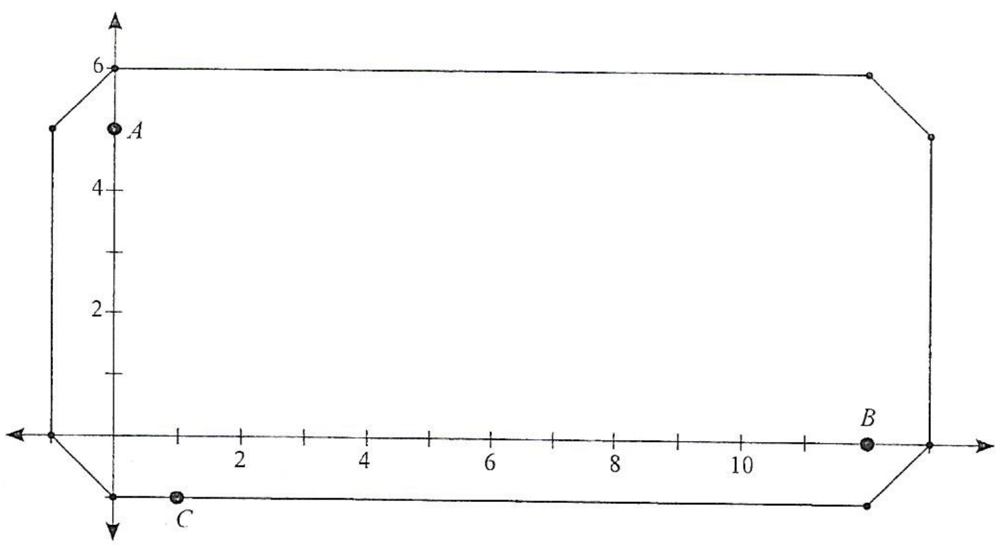
Similarly, $d(A,P) + d(P,B) = 19$ whenever P is on the segment from $(-1,0)$ to $(-1,5)$.
Finally, if P is on the segment from $(12,-1)$ to $(13,0)$, then $d(A,P) + d(B,P)$ is constant:

$$\begin{aligned} d(A,P) + d(B,P) &= |0-x| + |5-y| + |12-x| + |0-y| \\ &= x + (5-y) + (x-12) + (-y) \\ &= 2x - 2y - 7, \end{aligned}$$

and because the line segment has equation $x - y = 13$, this expression reduces to

$$\begin{aligned} d(A,P) + d(B,P) &= 2(x-y) - 7 \\ &= 2(13) - 7 \\ &= 19. \end{aligned}$$

Similarly, $d(A,P) + d(B,P) = 19$ on the segments joining $(13,5)$ and $(12,6)$, $(0,6)$ and $(-1,5)$, and $(-1,0)$ to $(0,-1)$. The shape of the "ellipse" is given below.



The simplest way to compute the polygon's area is to subtract the areas of the four corner triangles from that of the enclosing rectangle. The enclosing rectangle's area is $14 \cdot 7 = 98$, while each triangle has area $\frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$. Thus the area is $98 - 4 \cdot \frac{1}{2} = \mathbf{96}$.

Problem 7. The function f satisfies the relation $f(n) = f(n-1)f(n-2)$ for all integers n, and $f(n) > 0$ for all positive integers n. If $f(1) = \frac{f(2)}{512}$ and $\frac{1}{f(1)} = 2f(2)$, compute $f(f(4))$.

Solution 7. Substituting yields $\frac{512}{f(2)} = 2f(2) \Rightarrow (f(2))^2 = 256 \Rightarrow f(2) = 16$. Therefore $f(1) = \frac{1}{32}$. Using the recursion, $f(3) = \frac{1}{2}$ and $f(4) = 8$. So $f(f(4)) = f(8)$. Continue to apply the recursion:

$$f(5) = 4, \quad f(6) = 32, \quad f(7) = 128, \quad f(8) = \mathbf{4096}.$$

Alternate Solution: Let $g(n) = \log_2 f(n)$. Then $g(n) = g(n-1) + g(n-2)$, with initial conditions $g(1) = g(2) - 9$ and $-g(1) = 1 + g(2)$. From this, $g(1) = -5$ and $g(2) = 4$, and from the recursion,

$$g(3) = -1, \quad g(4) = 3,$$

so $f(4) = 2^{g(4)} = 8$. Continue to apply the recursion:

$$g(5) = 2, \quad g(6) = 5, \quad g(7) = 7, \quad g(8) = 12.$$

Because $g(f(4)) = 12$, it follows that $f(f(4)) = 2^{12} = \mathbf{4096}$.

Problem 8. Frank Narf accidentally read a degree n polynomial with integer coefficients backwards. That is, he read $a_n x^n + \ldots + a_1 x + a_0$ as $a_0 x^n + \ldots + a_{n-1}x + a_n$. Luckily, the reversed polynomial had the same zeros as the original polynomial. All the reversed polynomial's zeros were real, and also integers. If $1 \le n \le 7$, compute the number of such polynomials such that $\mathrm{GCD}(a_0, a_1, \ldots, a_n) = 1$.

Solution 8. When the coefficients of a polynomial f are reversed to form a new polynomial g, the zeros of g are the reciprocals of the zeros of f: r is a zero of f if and only if r^{-1} is a zero of g. In this case, the two polynomials have the *same* zeros; that is, whenever r is a zero of either, so must be r^{-1}. Furthermore, both r and r^{-1} must be real as well as integers, so $r = \pm 1$. As the only zeros are ± 1, and the greatest common divisor of all the coefficients is 1, the polynomial must have leading coefficient 1 or -1.

Thus

$$\begin{aligned} f(x) &= \pm(x \pm 1)(x \pm 1)\cdots(x \pm 1) \\ &= \pm(x+1)^k(x-1)^{n-k}. \end{aligned}$$

If A_n is the number of such degree n polynomials, then there are $n+1$ choices for k, $0 \le k \le n$. Thus $A_n = 2(n+1)$.

The number of such degree n polynomials for $1 \le n \le 7$ is the sum:

$$A_1 + A_2 + \ldots + A_7 = 2(2 + 3 + \ldots + 8) = 2 \cdot 35 = \mathbf{70}.$$

Problem 9. Given a regular 16-gon, extend three of its sides to form a triangle none of whose vertices lie on the 16-gon itself. Compute the number of noncongruent triangles that can be formed in this manner.

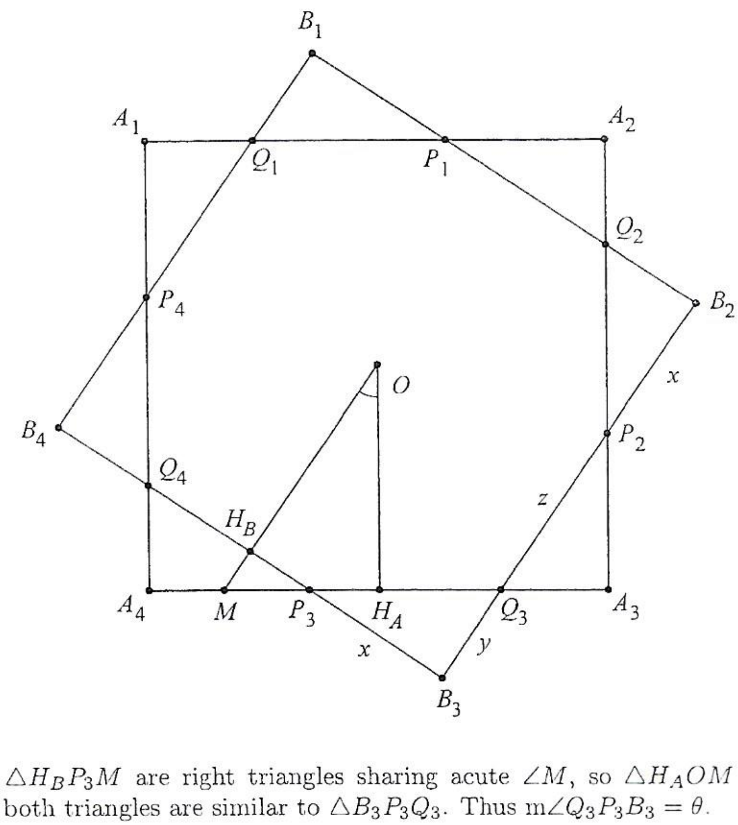
Solution 9. Label the sides of the polygon, in order, $s_0, s_1, \ldots, s_{15}$. First note that two sides of the polygon intersect at a vertex if and only if the sides are adjacent. So the sides chosen must be nonconsecutive. Second, if nonparallel sides s_i and s_j are extended, the angle of intersection is determined by $|i-j|$, as are the lengths of the extended portions of the segments. In other words, the *spacing* of the extended sides completely determines the shape of the triangle. So the problem reduces to selecting appropriate spacings, that is, finding integers $a, b, c \ge 2$ whose sum is 16. However, diametrically opposite sides are parallel, so (for example) the sides s_3 and s_{11} cannot both be used. Thus none of a, b, c may equal 8. Taking s_0 as the first side, the second side would be $s_{0+a} = s_a$, and the third side would be s_{a+b}, with c sides between s_{a+b} and s_0. To eliminate reflections and rotations, specify additionally that $a \ge b \ge c$. The allowable partitions are in the table below.

a	b	c	triangle
12	2	2	$s_0s_{12}s_{14}$
11	3	2	$s_0s_{11}s_{14}$
10	4	2	$s_0s_{10}s_{14}$
10	3	3	$s_0s_{10}s_{13}$
9	5	2	$s_0s_9s_{14}$
9	4	3	$s_0s_9s_{13}$
7	7	2	$s_0s_7s_{14}$
7	6	3	$s_0s_7s_{13}$
7	5	4	$s_0s_7s_{12}$
6	6	4	$s_0s_6s_{12}$
6	5	5	$s_0s_6s_{11}$

Thus there are **11** distinct such triangles.

Problem 10. Two square tiles of area 9 are placed with one directly on top of the other. The top tile is then rotated about its center by an acute angle θ. If the area of the overlapping region is 8, compute $\sin\theta + \cos\theta$.

Solution 10. In the diagram below, O is the center of both squares $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. Let P_1, P_2, P_3, P_4 and Q_1, Q_2, Q_3, Q_4 be the intersections of the sides of the squares as shown. Let H_A be on $\overline{A_3A_4}$ so that $\angle A_3H_AO$ is right. Similarly, let H_B be on $\overline{B_3B_4}$ such that $\angle B_3H_BO$ is right. Then the angle by which $B_1B_2B_3B_4$ was rotated is $\angle H_AOH_B$. Extend $\overline{OH_B}$ to meet $\overline{A_3A_4}$ at M.



Both $\triangle H_AOM$ and $\triangle H_BP_3M$ are right triangles sharing acute $\angle M$, so $\triangle H_AOM \sim \triangle H_BP_3M$. By an analogous argument, both triangles are similar to $\triangle B_3P_3Q_3$. Thus $m\angle Q_3P_3B_3 = \theta$.

Now let $B_3P_3 = x, B_3Q_3 = y$, and $P_3Q_3 = z$. By symmetry, notice that $B_3P_3 = B_2P_2$ and that $P_3Q_3 = P_2Q_3$. Thus

$$x + y + z = B_3Q_3 + Q_3P_2 + P_2B_2 = B_2B_3 = 3.$$

By the Pythagorean Theorem, $x^2 + y^2 = z^2$. Therefore

$$\begin{aligned} x + y &= 3 - z \\ x^2 + y^2 + 2xy &= 9 - 6z + z^2 \\ 2xy &= 9 - 6z. \end{aligned}$$

The value of xy can be determined from the areas of the four triangles $\triangle B_i P_i Q_i$. By symmetry, these four triangles are congruent to each other. Their total area is the area not in both squares, i.e., $9 - 8 = 1$. Thus $\frac{xy}{2} = \frac{1}{4}$, so $2xy = 1$.

Applying this result to the above equation,

$$\begin{aligned} 1 &= 9 - 6z \\ z &= \frac{4}{3}. \end{aligned}$$

The desired quantity is $\sin\theta + \cos\theta = \frac{x}{z} + \frac{y}{z}$, and

$$\begin{aligned} \frac{x}{z} + \frac{y}{z} &= \frac{x + y + z}{z} - \frac{z}{z} \\ &= \frac{3}{z} - 1 \\ &= \frac{5}{4}. \end{aligned}$$

NYSML 2000 POWER ROUND

60 Minutes

Any prime p other than 2 or 5 has a reciprocal which is a repeating decimal with no initial non-repeating segment. For prime p the number of digits in the repeating pattern of $1/p$ is called the period of $1/p$. The pattern itself will be called the cycle of $1/p$ and will be written as $1/p = =.\overline{a_1a_2...a_k}$ or sometimes as $a_1a_2 \cdots a_k$ where the a_i are digits. For example $1/37 = .\overline{027}$ so its period is 3 and its cycle is 027. Throughout this problem p will refer to a prime other than 2 or 5.

Modular Arithmetic

You may find the concept of modular arithmetic useful in this problem. We say "a is congruent to b mod m" and write $a \equiv b \pmod{m}$ if $a - b$ is a multiple of m. Thus $14 \equiv 6 \pmod 4$ since $14 - 6 = 8$ is a multiple of 4. Two numbers are congruent (mod m) if they give the same remainder upon division by m. If $a \equiv 0 \pmod{m}$ then $a - 0 = a$ is a multiple of m and we write $m|a$ and say "m divides a". Three properties of modular arithmetic that you may use without proof are:

M1 If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $a \pm c \equiv b \pm d \pmod{m}$.

M2 If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $ac \equiv bd \pmod{m}$.

M3 If $ab \equiv ac \pmod{m}$, then $b \equiv c \left(\text{mod} \dfrac{m}{\gcd(a,m)}\right)$, where $\gcd(a, m)$ is the greatest common divisor of a and m.

As an example of M3 note that $4 \cdot 5 \equiv 4 \cdot 8 \pmod 6$ but $5 \not\equiv 8 \pmod 6$. However $5 \equiv 8 \pmod{6/2 = 3}$, where $\gcd(4, 6) = 2$.

Fermat's Theorem

You will probably need to use Fermat's Theorem at some point during this question. It states that for any prime p which is not a factor of 10, $10^{p-1} \equiv 1 \pmod{p}$.

A Division Example

Observe the division at right carefully. It shows the determination of the cycle for 1/37, and the remainders are in boldface type. Notice that the first remainder is 1, even though you would not normally write it out this way. This just tells you that $10^0 \equiv 1 \pmod{37}$. The cycle starts to repeat as soon as the remainder of 1 occurs again.

```
      0.027
 37 )1.000
     0
     10
      0
     100
      74
      260
      259
        1
```

In this nine-part problem you may use any part at any time in a computation. However in the proof of any part you may only use and assume the truth of previous parts. The point value for each part is given in brackets.

1. Prove that $1/p$ has period k where k is the smallest positive integer such that $10^k \equiv 1 \pmod{p}$. [3]

2. There is exactly one prime p with period 4. Compute it. [Hint: Note that $10^k \equiv 1 \pmod{p}$, by M1, is equivalent to $10^k - 1 \equiv 0 \pmod{p}$ or $p|10^k - 1$.] [3]

3. Find the two primes p such that $1/p$ has period 6. [2]

4. Find all primes p such that $1/p$ has period 8. [3]

5. Prove that n/p has the same period as $1/p$ for $n = 2, 3, \cdots, p - 1$. [3]

6. Prove that if $1/p$ has period k then $k|p - 1$. [4]

7. Prove that the cycle of n/p is a cyclic permutation of the cycle for $1/p$ if and only if n occurs as a remainder in the division $1 \div p$. [4]

8. Compute the cycle for 1/29, given that it has period 28. [4]

9. The cycle for 1/7 is 142857. Notice that it can be divided into two sections, 142 and 857, in which corresponding entries add to 9 (1 + 8, 4 +5, 2 + 7). Prove that this is always the case. That is, let $1/p$ have an even period $2k$. Prove that the cycle for $1/p$ can be written as $a_1a_2 \cdots a_kb_1b_2 \cdots b_k$, where $a_i + b_i = 9$ for $i = 1, 2, \cdots, k$. [4]

NYSML 2000 SOLUTIONS - POWER ROUND

1. In the long division $1 \div p$, the first remainder is 1. Since the remainder determines the next quotient, which determines the next remainder etc., the digits in the cycle will start to repeat as soon as a remainder of 1 occurs again. This must happen, since by Fermat's Theorem, $10^{p-1} \equiv 1 \pmod{p}$. Suppose this first happens in the kth decimal place. Then it will happen in the units place in the division $10^k \div p$, which means that $10^k \equiv 1 \pmod{p}$. Further, this will happen for no smaller positive power of 10. This proves the result.

2. The prime factorization of $10^4 - 1 = 9999$ is $9999 = 3^2 \cdot 11 \cdot 101$. Thus the only possible primes whose reciprocals have period 4 are 3, 11, and 101. Since 3 and 11 have periods 1 and 2 respectively and 101 does indeed have period 4, it is the only such prime.

3. The problem reduces to finding the prime factors of 111111. Since 3 and 11 are clearly factors, divide by 33 to obtain the quotient 3367. By trial and error we find that $3367 = 7 \cdot 13 \cdot 37$. We have seen that 1/37 has period 3. Since 1/7 and 1/13 do have period 6, 7 and 13 are the only such primes.

4. Since 11 is clearly a factor of 11111111, divide by it to obtain the quotient 1010101. Try dividing this by 101. The result is 10001. By trial and error find that $10001 = 73 \cdot 137$, so $11111111 = 11 \cdot 73 \cdot 101 \cdot 137$. Checking on a calculator shows that of the four possibilities only 73 and 137 have reciprocals with period 8.

5. Similar to the discussion in Part 1, the cycle will repeat as soon as the first remainder, which in this case is n, occurs again in the division $n \div p$. This will occur for the smallest positive integer k for which $n \cdot 10^k \equiv n \pmod{p}$. Since $\gcd(n, p) = 1$, by M3 we may divide both sides of the congruence by n to obtain $10^k \equiv 1 \pmod{p}$. But this value of k is the period of p.

6. By the division algorithm, there are integers m and r such that $p - 1 = mk + r$, for quotient m and remainder r, with $0 \le r \le k - 1$. By Fermat's Theorem, $10^{p-1} \equiv 1 \pmod{p}$. Thus $10^{p-1} = 10^{mk+r} = 10^{mk} \cdot 10^r = (10^k)^m \cdot 10^r \equiv 1 \pmod{p}$. But $10^k \equiv 1 \pmod{p}$ so, by M2, $(10^k)^m \equiv 1^m = 1 \pmod{p}$. Therefore $10^r \equiv 1 \pmod{p}$, and since k is the smallest positive integer for which this is true it follows that $r = 0$ and $k | p - 1$.

7. The cycle for any fraction is determined by the remainders that occur in the division. Thus suppose that n occurs as a remainder in the division $1 \div p$. Since n will occur as the first remainder in the division $n \div p$, the cycle for n/p will "pick up" the cycle for $1/p$ but starting at a different digit. Now suppose that the cycle for n/p is not a cyclic permutation of the cycle for $1/p$. If n occurred as a remainder in the division $n \div p$, as seen above, the cycle would "pick up" the cycle for $1/p$. Since, by hypothesis, this does not happen, n does not occur as a remainder.

8. These calculations were done on a TI-85 calculator. Note that by using Part 9 we only need to find the first 14 digits in the cycle for 1/29 since the rest can be found by subtraction from 9. Now for 1/29 the calculator returns .034482758621. The 1 in the last place may be a result of rounding up so we discard it and now know the first 10 digits in the cycle. Using Part 7, we perform the long division and find that the first five remainders are 1, 10, 13, 14, and 24. Thus the cycle for 24/29 is a cyclic permutation of the cycle for 1/29. The calculator returns .827586206897. Discarding the 7 and combining the overlapping portions of the cycle gives .03448275862068|9, where the bar marks the midpoint of the cycle. The remaining digits can easily be determined to get the full cycle $1/29 = \overline{.03448275862068|96551724137931}$.
A method of solution that does not involve Part 9 is to calculate various divisions $n/29$ knowing that

the results are all segments of the cycle. After enough such divisions are performed the whole cycle can be pieced together.

9. Let $1/p$ have period $2k$. Then we may write $1/p = .\overline{a_1a_2 \cdots a_k b_1 b_2 \cdots b_k}$. We must show that $a_i + b_i = 9$ for each $i = 1, 2, \cdots, k$. Notice that since $1/p$ cannot be an integer the possibilities $a_i = b_i = 0$ and $a_i = b_i = 9$ for $i = 1, 2, \cdots, k$ are ruled out. By definition of period, $p | 10^{2k} - 1 = (10^k + 1)(10^k - 1)$.

Also by definition of period, $p \nmid 10^k - 1$ so $p | 10^k + 1$. Now $1/p = .\overline{a_1a_2 \cdots a_k b_1 b_2 \cdots b_k}$ so $10^k/p = a_1a_2 \cdots a_k.\overline{b_1b_2 \cdots b_k a_1 a_2 \cdots a_k}$. Thus

$$\frac{10^k}{p} + \frac{1}{p} = \frac{10^k + 1}{p} = a_1a_2 \ldots a_k.\overline{b_1b_2 \ldots b_k a_1 a_2 \ldots a_k} + .\overline{a_1a_2 \ldots a_k b_1 b_2 \ldots b_k}$$

Since $p | 10^k + 1$, the expression after the second equal sign must be an integer and thus the decimal parts must add to 1. This will happen if $a_i + b_i = 9$ for $i = 1, 2, \cdots, k$. By uniqueness of decimal expansions (the ambiguous cases have been ruled out) it must be the case that $ai + bi = 9$ for $i = 1, 2, \cdots, k$.

This result is called Midy's Theorem. It can also be shown that if $1/p$ has period $2k$ the cycle of remainders can be written as $r_1, r_2, \cdots, r_k, s_1, s_2, \cdots, s_k$ where $r_i + s_i = p$ for $i = 1, 2, \cdots, k$.

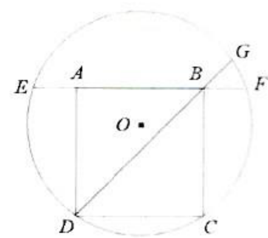
R1-1 Let **b** and **h** denote the base and height of a triangle whose area is 200.
Compute the smallest value of $b + h$. Pass back your answer.

R1-2 Let T = TNYWR. Compute the value of x that satisfies $\log_2(T-8) = 2 + \log_4(T+x)$.
Pass back your answer.

R1-3 Let T = TNYWR. And set $K = 2[T]$ where $[x]$ denotes the greatest integer function. If $f(n) = i^n + \frac{1}{i^n}$ where $i = \sqrt{-1}$, compute the number of integers n, $1 \le n \le 2005$, such that $f(K) + f(n) = 0$.
Hand in your answer.

R2-1 Compute the value of $x + y$ if $\frac{x^2}{2005} - \frac{y^2}{2005} = 1$ and $\frac{x}{2005} - \frac{y}{2005} = \frac{1}{3}$.
Hand in your answer.

R2-2 Let T = TNYWR. In the figure, *ABCD* is a square, *C* and *D* lie on circle *O*, Chord $\overline{EF}$ has length 10, and $AB = 2T$. $\overline{DB}$ meets circle *O* at *G*. Compute $\frac{1}{BG}$.
Pass back your answer.



R2-3 Let T = TNYWR. Compute the value of $\left(\cos\left(\sin^{-1} T\right)\right)^2 + T^2$.
Pass back your answer.

R3-1 Compute the smallest of 37 consecutive integers if their sum is 1665. Pass back your answer.

R3-2 Let T = TNYWR. Compute the smallest two-digit integer such that increasing it by *T* yields the two-digit number with its digits reversed. Pass back your answer.

R3-3 Let T = TNYWR. Compute the number os sides of an equiangular polygon if the measure of each interior angle is *T* times the measure of the exterior angle. Pass in your answer.

ARML Super Relay – 2005

1. Let b and h denote the base and height of a triangle whose area is 200. Compute the smallest value of $b+h$.

2. Let T = TNYWR. Compute the value of x that satisfies $\log_2(T-8) = 2 + \log_4(T+x)$.

3. Let T = TNYWR and set $K = 2[T]$ where $[x]$ denotes the greatest integer function. If $f(n) = i^n + \dfrac{1}{i^n}$ where $i = \sqrt{-1}$, compute the number of integers n, $1 \le n \le 2005$, such that $f(K) + f(n) = 0$.

Solutions to the ARML Super Relay – 2005

1. $\frac{1}{2}bh = 200 \rightarrow bh = 400$. By the AM-GM inequality, $\frac{b+h}{2} \ge \sqrt{bh} \rightarrow b+h \ge 2\sqrt{400} = \boxed{40}$.

2. Since $\log_{a^2} b = \frac{1}{2}\log_a b$, then $\log_2(T-8) = 2 + \log_4(T+x) = 2 + \frac{1}{2}\log_2(T+x) \rightarrow$

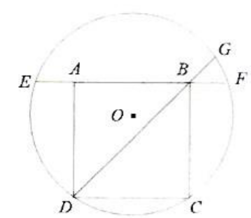
$2\log_2(T-8) = 4 + \log_2(T+x) \rightarrow \log_2(T-8)^2 - \log_2(T+x) = 4 \rightarrow \log_2\left(\frac{(T-8)^2}{T+x}\right) = 4 \rightarrow$

$(T-8)^2 = 2^4 \cdot (T+x) \rightarrow x = \frac{(T-8)^2}{16} - T$. Since $T = 40$, $x = \frac{32^2}{16} - 40 = \boxed{24}$.

3. Note that if $n = 0 \pmod 4$, $f(n) = 2$; if $n = 1 \pmod 4$, $f(n) = 0$; if $n = 2 \pmod 4$, $f(n) = -2$, and if $n = 3 \pmod 4$, $f(n) = 0$. Since $K = 2[T]$, K is even. If $K = 0 \pmod 4$, then n must equal $2 \pmod 4$, i.e., $n = 2, 6, \ldots, 2002$. There are 501 such numbers. If $K = 2 \pmod 4$, then n must equal $0 \pmod 4$, i.e., $n = 4, 8, \ldots, 2004$. There are also 501 such numbers. So, regardless of T, there are $\boxed{501}$ values of n.

15. Compute the value of $x+y$ if $\frac{x^2}{2005} - \frac{y^2}{2005} = 1$ and $\frac{x}{2005} - \frac{y}{2005} = \frac{1}{3}$.

14. Let T = TNYWR. In the figure, $ABCD$ is a square, C and D lie on circle O, chord $\overline{EF}$ has length 10, and $AB = 2T$. $\overrightarrow{DB}$ meets circle O at G. Compute $\frac{1}{BG}$.



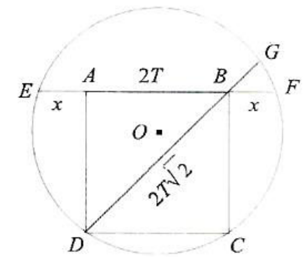
13. Let T = TNYWR. Compute the value of $\left(\cos\left(\sin^{-1} T\right)\right)^2 + T^2$.

15. $\frac{1}{2005}(x-y)(x+y)=1$ and $\frac{1}{2005}(x-y)=\frac{1}{3} \rightarrow \frac{1}{3}(x+y)=1$. Thus, $x+y = \boxed{3}$.

14. Let $AE = BF = x$. Then $(2T+x)x = (2T\sqrt{2})(BG) \rightarrow$

$\frac{1}{BG} = \frac{2T\sqrt{2}}{(2T+x)x}$. Since $2x + 2T = 10$, $x = 5 - T$, making

$\frac{1}{BG} = \frac{2T\sqrt{2}}{(2T+5-T)(5-T)} = \frac{2T\sqrt{2}}{25-T^2}$. $T = 3 \rightarrow \frac{1}{BG} = \boxed{\frac{3\sqrt{2}}{8}}$.



13. From $\sin^{-1}T = \theta$, we obtain $\sin\theta = T \rightarrow \cos\theta = \sqrt{1-T^2}$. It is positive because $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. So,

$\left(\cos(\sin^{-1}T)\right)^2 + T^2 = (\cos\theta)^2 + T^2 = 1 - T^2 + T^2 = \boxed{1}$. As long as $-1 \le T \le 1$, T is irrelevant.

R3-1 Compute the smallest of 37 consecutive integers if their sum is 1665. Pass back your answer.

R3-2 Let T = TNYWR. Compute the smallest two-digit integer such that increasing it by *T* yields the two-digit number with its digits reversed. Pass back your answer.

R3-3 Let T = TNYWR. Compute the number os sides of an equiangular polygon if the measure of each interior angle is *T* times the measure of the exterior angle. Pass in your answer.

Note, When I retyped the questions, I used T instead of N so that T = TNYWR.

NYSML 1995 – SOLUTIONS - Relay Round

Relay 1

R1-1. **Answer:** 27. The median number is 1665/37, or 45. Since there are 18 consecutive integers on each side of 45, the smallest integer is 45 – 18, or 27.

R1-2. **Answer:** 14. If 10t+u represents the original number, then 10u+t represents the reversal. Thus, 10t+u+N = 10u+t. This yields 9u–9t = N, or u–t = $\frac{N}{5}$ (So N must be a multiple of 9. Hmm). Since N = 27, u - t = 3. Now, list all possibilities: 14, 25, 36, 47, 58, and 69. The smallest is 14.

R1-3. **Answer:** 30. In any polygon an interior angle and its exterior angle are supplementary (sketch it!). Let the exterior angle contain *e* degrees. Then its interior angle contains N*e* degrees, and we have N*e* + *e* = 180. Thus, (N + 1)*e* = 180, or $e = \frac{180}{N+1}$. BUT: we want *n*, the number of sides. Since $e = \frac{360}{n}$ in any equiangular polygon, we have $\frac{360}{n} = \frac{180}{N+1}$ or $n = 2N + 2$. Since N = 14, $n = 30$.